Global Dynamics at the Zero Lower Bound*

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ABSTRACT

This article presents global solutions to standard New Keynesian models with a zero lower bound (ZLB) constraint on the nominal interest rate. We provide the solution for all combinations of technology and discount factor shocks and a thorough explanation of dynamics. We initially focus on the New Keynesian model without capital (Model 1) but then study the model with capital (Model 2). Capital adds another mechanism for intertemporal substitution, which strengthens the expectational effects of the ZLB. We use these models to analyze why technology shocks at the ZLB may have unconventional effects, the likelihood of ZLB events, and the tradeoffs faced by the central bank under a dual mandate. Three main findings emerge: (1) In Model 1, the output gap specification in the Taylor rule may reverse the effects of technology shocks at the ZLB; (2) When the central bank targets the steady-state output gap in Model 2, a positive technology shock at the ZLB leads to more pronounced unconventional dynamics than in Model 1; (3) The constrained linear model provides a decent approximation of the nonlinear model without capital, but meaningful differences exist between the solutions in Model 2.

Keywords: Monetary Policy; Zero Lower Bound; Global Solution Method

JEL Classifications: E31; E42; E58; E61

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1 INTRODUCTION

In the aftermath of the financial crisis, aggregate demand fell sharply. The Fed quickly responded by lowering its policy rate to its zero lower bound (ZLB) by the end of 2008. Five years after the crisis began, the Fed’s target interest rate remains near zero and the economy is below potential.

Figure 1 shows the U.S. and Japanese interbank lending rate and employment-to-population percentage from 1992-2013. The U.S. policy rate (solid line) has varied between 6.5 percent and 0 since 1992 and has been held below 25 basis points since the end of 2008. During this period, policymakers shifted their focus from inflation to the real economy, since the inflation rate has been at or below the Fed’s inflation target. The Bank of Japan sharply lowered its policy rate in 1992 (dashed line), reaching 50 basis points in 1995. Since then it has remained between 0 and 50 basis points, while the employment-to-population percentage steadily fell from 62 percent to about 57.5 percent. The Japanese economy slightly rebounded in the mid-2000s, but following the financial crisis, the policy rate was cut and the employment-to-population percentage fell further.

Over the last two decades, the Japanese economy has endured anemic growth in real GDP and slight deflation. Their experience generated a significant amount of research on the effects of the Bank of Japan’s zero interest rate policy [Braun and Waki (2006); Eggertsson and Woodford (2003); Hoshi and Kashyap (2000); Ito and Mishkin (2006); Krugman (1998); Posen (1998)]. Many arguments for avoiding the ZLB are motivated, in part, by the recent Japanese experience.

Our paper focuses on both discount factor shocks and technology shocks, whereas many other papers consider only discount factor shocks. While no one believes nominal interest rates fell to zero in December 2008 due to a series of positive technology shocks, we examine them because we want to show how the economy reacts to technology shocks when a discount factor shock moves the nominal interest rate to the ZLB. Furthermore, Gust et al. (2013) estimate that the level of technology was 0.5 percent above steady state in the second quarter of 2009.

We initially focus on the New Keynesian model without capital, but then study the model with capital to draw comparisons. In the model without capital, positive technology shocks may have unconventional effects at the ZLB. We find that whether those unconventional effects occur...
depends on how the central bank reacts to the output gap when the ZLB does not bind.\footnote{Wieland (2014) uses structural VAR evidence to argue that these unconventional dynamics did not occur following the 2011 earthquake/tsunami in Japan or the recent oil supply shocks. Braun and Waki (2006) show that technology shocks yield unconventional dynamics at the ZLB in a log-linearized model with capital accumulation where monetary policy responds to a steady-state output gap. Using a nonlinear model with capital, Braun and Körber (2011) show that these unconventional dynamics may disappear if the expected duration at the ZLB is short enough.} When the central bank aggressively targets the deviation of output from its steady state, positive technology shocks have unconventional effects on the economy at the ZLB, which become more pronounced as the shock increases. Those unconventional dynamics, however, disappear when the central bank targets the deviation of output from its potential, where potential output equals the output produced in our model with flexible prices. In this case, even large shocks, which generate long ZLB events, do not reduce output at the ZLB. If the central bank does not respond to either output gap, the unconventional dynamics partially disappear, as only large shocks reduce output at the ZLB.

We focus on the case where the output gap is defined as the deviation of output from its steady state because policymakers, in the short-to-medium term, assume potential output grows at a relatively constant rate [Basu and Fernald (2009)]. Potential output measures are revised in the long run following incoming information about shocks, but the revisions occur well after the temporary economic effects from sticky prices have dissipated. Orphanides and van Norden (2002) and Orphanides (2003a, b) document that historically neither the Fed nor standard statistical methods have been able to detect changes in potential output until well after they have occurred.

Much of the literature on the ZLB uses models without capital.\footnote{A notable exception is Christiano (2004), which generalizes Eggertsson and Woodford (2003) to include capital.} Capital accumulation is an important feature because it gives households another margin to smooth consumption, which strengthens the expectational effects of the ZLB and impacts dynamics. Arbitrage implies that the real interest rate equals the expected future real rental rate of capital. The large decline in demand when the ZLB binds leads to a sharp reduction in the rental rate of capital. Thus, the household places increasing weight on a lower rental rate as the ZLB nears, which pushes down the real interest rate even in states where the ZLB does not bind. We include capital adjustment costs to dampen the volatility of investment. Specifically, capital adjustment costs make investment less attractive as a consumption smoothing mechanism, which causes a greater reduction in consumption and a larger increase in the real interest rate at the ZLB. When the central bank targets the steady-state output gap in our model with capital, a positive technology shock at the ZLB leads to more pronounced unconventional dynamics than in the model without capital.

We also evaluate how alternative versions of the Taylor rule affect the likelihood of encountering the ZLB and the efficacy of stabilization policy. A policy rule based on a dual mandate is more likely to cause ZLB events when the central bank places greater emphasis on the steady-state output gap in our model without capital. The opposite result occurs when the central bank emphasizes the potential output gap.\footnote{Adam and Billi (2006) find that it is optimal to reduce the nominal interest rate more aggressively in response to adverse shocks when the central bank is constrained by the ZLB, despite the welfare consequences that occur when the ZLB is hit. Several papers discuss optimal policy with a ZLB constraint and provide analysis of the welfare losses at the ZLB [Eggertsson and Woodford (2003); Günter et al. (2004); Jung et al. (2005); Nakov (2008); Werning (2011)].} When we compare our two models with only discount factor shocks, we find that a larger policy response to the steady-state output gap decreases the frequency of ZLB events in our model without capital, but increases the frequency in our model with capital.

Any ZLB analysis is complicated by the occasionally binding constraint on the monetary policy rule, which imposes a discontinuity in the policy functions. The literature has employed a variety
of techniques to address this problem. Many studies linearize the equilibrium system, except the monetary policy rule, and solve either the deterministic model or the stochastic model based on specific sequences of shocks [Christiano et al. (2011); Eggertsson and Woodford (2003); Gertler and Karadi (2011)]. In these setups, the duration of the ZLB event is predetermined. Extensions of this work allow for stochastic ZLB events, but do not allow for recurring ZLB events [Braun and Waki (2006); Erceg and Linde (2010)]. Braun and Körber (2011) solve the nonlinear model, but use an extended shooting algorithm that still requires strong assumptions about future shocks.

There are three main drawbacks with these solution techniques. First, they violate the Lucas (1976) critique, which says that if policy changes, it is important to account for changes in expectations when studying the effects of the new policy. The sequences of shocks often used are very low probability events. Thus, when the ZLB is hit or continues to bind for several periods, the policy is virtually unaccounted for in the household’s expectations. This has important implications for determinacy and dynamics [Richter and Throckmorton (2013)]. Second, using log-linearized models creates the potential for large approximation errors. Braun et al. (2012) and Fernández-Villaverde et al. (2012) provide examples of the mistakes resulting from linearized models without capital evaluated at the ZLB. Moreover, Braun et al. (2012) argue that linearized models often lead to incorrect inferences about existence of equilibrium, uniqueness, and local dynamics. We find that the constrained linear model provides a fairly good approximation of the nonlinear model without capital, but the error is much larger in a model with capital.4 As a consequence, the simulated moments and model predictions are drastically different in the linearized model with capital. Third, these methods prohibit Monte Carlo simulations of the model, which are necessary to study the conditional and unconditional probability distributions across alternative model specifications.

Our paper avoids these problems by solving for the global nonlinear solution to standard New Keynesian models that include an occasionally binding ZLB constraint on the nominal interest rate in the monetary policy rule.5 Rather than focus on specific sequences of shocks, we provide the solution for all combinations of discount factor and technology shocks and a thorough explanation of how dynamics change across the entire state space. Our solution method emphasizes accuracy to capture important expectational effects of going to and returning from the ZLB, which commonly used solution methods based on specific sequences of shocks cannot capture.

The remainder of the paper proceeds as follows. Section 2 describes the two alternative models. Section 3 describes the calibration and solution procedure, and sections 4 through 6 present the results. These sections report the model solutions across all technology and discount factor shocks, the dynamics at the ZLB, and the likelihood of hitting the ZLB. We also explain how the monetary policy rule impacts these results and provide a detailed comparison between the New Keynesian models with and without capital. Lastly, we present new evidence that the solutions to the constrained linear and nonlinear models are significantly different. Section 7 concludes.

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4Braun and Waki (2010) show that the approximation error in a perfect-foresight version of a linear model with capital where monetary policy does not respond to an output gap overstates the government spending multiplier.

5Recent papers that study the ZLB using global nonlinear solutions include Wolman (2005), Basu and Bundick (2012), Fernández-Villaverde et al. (2012), Nakata (2012), Aruoba and Schorfheide (2013), Gust et al. (2013), and Mertens and Ravn (2013). Fernández-Villaverde et al. (2012) calculates the conditional and unconditional moments of ZLB events. Wolman (2005) shows that the real effects of the ZLB depend on the policy rule and nominal rigidities. Gust et al. (2013) estimates the extent to which the ZLB constrained the central bank’s ability to stabilize the economy. Aruoba and Schorfheide (2013) and Mertens and Ravn (2013) show how the ZLB affects fiscal multipliers and Basu and Bundick (2012) and Nakata (2012) show that the ZLB magnifies the effect of uncertainty on aggregate demand.
2 ECONOMIC MODELS

This section presents two New Keynesian models with Rotemberg (1982) price adjustment costs. Both models assume stochastic processes for the discount factor and technology, but they differ in their treatment of capital. That is, Model 1 does not include capital while Model 2 does.

2.1 MODEL 1: BASELINE A representative household chooses \( \{c_t, n_t, b_t\}_{t=0}^{\infty} \) to maximize expected lifetime utility, given by, 
\[
E_0 \sum_{t=0}^{\infty} \bar{\beta} \{ \log c_t - \chi n_t^{1+\eta}/(1 + \eta) \},
\]
where \( 1/\eta \) is the Frisch elasticity of labor supply, \( c_t \) is consumption of the final good, \( n_t \) is labor hours, \( E_0 \) is an expectation operator conditional on information available in period 0, \( \bar{\beta}_0 = 1, \) and \( \beta_t = \prod_{j=1}^t \beta_j \) for \( t > 0. \) \( \beta_j \) is a time-varying subjective discount factor that evolves according to
\[
\beta_j = \bar{\beta}((\beta_{j-1}/\bar{\beta}))^{\rho_j} \exp(\varepsilon_{\beta,j}), \tag{1}
\]
where \( \bar{\beta} \) is the steady-state discount factor, \( 0 \leq \rho_j < 1, \) and \( \varepsilon_{\beta,j} \sim N(0, \sigma_j^2). \) These choices are constrained by \( c_t + b_t = w_t n_t + r_{t-1} b_{t-1}/\pi_t + d_t, \) where \( \pi_t = p_t/p_{t-1} \) is the gross inflation rate, \( w_t \) is the real wage rate, \( b_t \) is a 1-period real bond, \( r_t \) is the gross nominal interest rate, and \( d_t \) are profits from intermediate firms. The optimality conditions to the household’s problem imply
\[
w_t = \chi n_t^{\eta} c_t, \tag{2}
\]
\[
1 = r_t E_t \{ \beta_{t+1} (c_t/c_{t+1})^\sigma / \pi_{t+1} \}. \tag{3}
\]

The production sector consists of monopolistically competitive intermediate goods firms who produce a continuum of differentiated inputs and a representative final goods firm. Each firm \( f \in [0, 1] \) in the intermediate goods sector produces a differentiated good, \( y_t(f), \) with identical technologies given by \( y_t(f) = z_t n_t(f), \) where \( n_t(f) \) is the level of employment used by firm \( f. \) \( z_t \) represents the level of technology, which is common across firms and follows
\[
z_t = \bar{z}(z_{t-1}/\bar{z})^{\rho_z} \exp(\varepsilon_{z,t}), \tag{4}
\]
where \( \bar{z} \) is steady-state technology, \( 0 \leq \rho_z < 1, \) and \( \varepsilon_{z,t} \sim N(0, \sigma_z^2). \) Each intermediate firm chooses its labor supply to minimize its operating costs, \( w_t n_t(f), \) subject to its production function.

Using a Dixit and Stiglitz (1977) aggregator, the representative final goods firm purchases \( y_t(f) \) units from each intermediate goods firm to produce the final good, \( y_t \equiv \int_0^1 y_t(f) (\theta - 1)/\theta df \int_0^{(1-\theta)/\theta} df, \) where \( \theta > 1 \) measures the elasticity of substitution between the intermediate goods. The final goods firm sells the final good to the household to maximize profits, which determines its demand for intermediate good \( f, \) given by, \( y_t(f) = (p_t(f)/p_t)^{-\theta} y_t, \) where \( p_t = [\int_0^1 p_t(f) (1-\theta)/\theta df]^{1/(1-\theta)} \) is the price of the final good. Following Rotemberg (1982), each firm faces a cost to adjusting its price, which emphasizes the potentially negative effect that price changes can have on customer-firm relationships. Using the functional form in Ireland (1997), real profits of firm \( f \) are
\[
d_t(f) = \left[ (p_t(f)/p_t)^{1-\theta} - \Psi_t \left( p_t(f)/p_t \right)^{-\theta} - \frac{\varphi}{2} \left( \frac{p_t(f)}{\bar{\pi} p_{t-1}(f)} - 1 \right)^2 \right] y_t,
\]
where \( \varphi \geq 0 \) determines the magnitude of the adjustment cost, \( \Psi_t = w_t/z_t \) is real marginal costs, and \( \bar{\pi} \) is the steady-state gross inflation rate. Each intermediate goods firm chooses its price level,
In this paper, the output gap is either defined as the deviation of output from its steady state, or as the deviation from potential output, or as the deviation from the firm’s markup of price over marginal cost.

Each period, the central bank sets the gross nominal interest rate according to

\[
    r_t = \max\{1, \bar{r} \frac{\bar{\pi}_t}{\pi^*} \phi_\pi(y_t/\bar{y}_t)^{\phi_y},
\]

where \( \pi^* = \bar{\pi} \) is the inflation rate target and \( \phi_\pi \) and \( \phi_y \) are the policy responses to inflation and output.⁶ In this paper, the output gap is either defined as the deviation of output from its steady state, \( y_t^* = \bar{y} \), or as the deviation from potential output, \( y_t^* = y_t^\pi = (\lambda\mu)^{-1/(1+\eta)} z_t \), where potential output equals the level of output when \( \bar{\pi} = 0 \). We also examine the case in which \( \phi_y = 0 \).

The resource constraint is given by

\[
    c_t = [1 - \phi(\bar{\pi}_t/\pi^* - 1)^2/2] y_t \equiv y_t^{adj},
\]

where \( y_t^{adj} \) includes the value added by intermediate firms, which is their output minus quadratic price adjustment costs. A competitive equilibrium consists of sequences of quantities \( \{c_t, n_t, b_t, y_t\}_{t=0}^{\infty} \), prices \( \{w_t, r_t, \pi_t\}_{t=0}^{\infty} \), and exogenous variables \( \{\beta_t, z_t\}_{t=0}^{\infty} \) that satisfy the household’s and firm’s optimality conditions [(2),(3),(5)], the production function, \( y_t = z_t n_t \), the monetary policy rule [(6)], the stochastic processes [(1),(4)], the bond market clearing condition, \( b_t = 0 \), and the resource constraint.

### 2.2 Model 2: Baseline with Capital

Model 2 adds capital accumulation to Model 1. The household chooses sequences \( \{c_t, i_t, n_t, b_t\}_{t=0}^{\infty} \) to maximize the preferences in Model 1 subject to

\[
    c_t + i_t + \Phi(i_t/k_{t-1})k_{t-1} + b_t = w_t n_t + r^k_t k_{t-1} + r_{t-1} b_{t-1}/\pi_t + d_t,
\]

where \( i_t \) is investment, \( k_t \) is the capital stock, \( r^k_t \) is the real capital rental rate, and \( \Phi(\cdot) \) is a positive, increasing, and convex function that measures the cost of adjusting the capital stock. We assume \( \Phi(x) = \phi(x - \delta)^2/2 \), where \( \phi \) measures the size of the adjustment cost. While other papers utilize alternative specifications of capital/investment adjustment costs, we use this specification because it does not add another state variable to our model, which allows us to present the complete model solution. Optimality yields an equation for Tobin’s \( q \) and a consumption Euler equation, given by,

\[
    q_t = 1 + \phi(i_t/k_{t-1} - \delta),
\]

\[
    q_t = E_t \left\{ \beta_{t+1} \frac{c_t}{c_{t+1}} \left( r^k_{t+1} - \frac{\phi}{2} \left( \frac{i_{t+1}}{k_t} - \delta \right)^2 + \phi \left( \frac{i_{t+1}}{k_t} - \delta \right) \frac{i_{t+1}}{k_t} + (1 - \delta) q_{t+1} \right) \right\}.
\]

⁶Although we set the lower bound on the policy rate equal to zero, the same dynamics would occur if the bound was set to a small but positive value. The key is the existence of a lower bound, which prevents the Fed from responding to adverse shocks to the economy. This is important because the Fed has not targeted a policy rate equal to zero.
Every intermediate firm then chooses its inputs who chose these parameters so that a discount factor shock \( h \) as a half life. The risk-free real interest rate is set to \( \rho_y \). The elasticity of substitution between intermediate goods, \( \varphi \), is calibrated to 59.11, which corresponds to an average markup of price over marginal cost equal to \( \phi \). The price adjustment parameter, \( \phi \), is set to 5.6, which follows Eberly (1997) and Erceg and Levin (2003). The elasticity of substitution between intermediate goods, \( \theta \), is set to 6, which corresponds to an average markup of price over marginal cost equal to 20 percent. The costly price adjustment parameter, \( \varphi \), is calibrated to 59.11, which is consistent with a Calvo (1983) price-setting specification where prices change on average once every four quarters. Steady-state technology, \( \bar{\varepsilon} \), is normalized to 1, and we set \( \rho_\varepsilon = 0.9 \) and \( \sigma_\varepsilon = 0.0025 \).

Each firm \( f \in [0,1] \) in the intermediate goods sector produces a differentiated good, \( y_t(f) \), with identical technologies given by \( y_t(f) = z_t k_{t-1}(f)^\alpha n_t(f)^{1-\alpha} \), where \( k_t(f) \) and \( n_t(f) \) are the levels of capital and employment used by firm \( i \). Every intermediate firm then chooses its inputs to minimize its operating costs, \( r_t^f k_{t-1}(f) + w_t n_t(f) \), subject to its production function, which yields a consolidated optimality condition, given by,

\[
\alpha w_t n_t = \beta (1 - \alpha) r_t^f k_{t-1}. \tag{11}
\]

The firm pricing equation, (5), remains unchanged, except that \( \Psi_t = w_t^{1-\alpha} (r_t^f)^\alpha / [z(1 - \alpha)^{1-\alpha} \alpha^\alpha] \).

The resource constraint is given by \( c_t + \delta_t + \Phi(i_t/k_{t-1}) = y_t \), which includes resources lost to both price and capital adjustment costs. A competitive equilibrium consists of sequences of quantities \( \{c_t, n_t, i_t, k_t, b_t, y_t\}_t \), prices \( \{w_t, r_t^f, r_t, \pi_t, q_t\}_t \), and exogenous variables \( \{\beta_t, z_t\}_t \) that satisfy the household’s and firm’s optimality conditions [(2),(3),(5),(9),(10),(11)], the production function, \( y_t = z_t k_{t-1}^\alpha n_t^{1-\alpha} \), the monetary policy rule [(6)], the stochastic processes [(1),(4)], the law of motion for capital [(7)], bond market clearing, \( b_t = 0 \), and the resource constraint.

### 3 Calibration and Solution Technique

The models in section 2 are calibrated at a quarterly frequency and the parameters are given in table 1. The risk-free real interest rate is set to 2 percent percent annually, which implies a steady-state quarterly discount factor, \( \beta \), equal to 0.995. We set the persistence of the discount factor, \( \rho_\beta \), equal to 0.8 and the standard deviation of the shock, \( \sigma_\beta \), equal to 0.0025. We follow Fernández-Villaverde et al. (2012) who chose these parameters so that a discount factor shock has a half life of about 3 quarters and an unconditional standard deviation of 0.42 percent. The Frisch elasticity of labor supply, \( 1/\eta \), is set to 3, which is consistent with estimates in Peterman (2012). The leisure preference parameter, \( \chi \), is calibrated so that steady-state labor equals 1/3 of the available time. Capital’s share of output, \( \alpha \), is set to 0.33 and the quarterly depreciation rate, \( \delta \), equals 2.5 percent.

The capital adjustment cost parameter, \( \phi \), is set to 5.6, which follows Eberly (1997) and Erceg and Levin (2003). The elasticity of substitution between intermediate goods, \( \theta \), is set to 6, which corresponds to an average markup of price over marginal cost equal to 20 percent. The costly price adjustment parameter, \( \varphi \), is calibrated to 59.11, which is consistent with a Calvo (1983) price-setting specification where prices change on average once every four quarters. Steady-state technology, \( \bar{\varepsilon} \), is normalized to 1, and we set \( \rho_\varepsilon = 0.9 \) and \( \sigma_\varepsilon = 0.0025 \).

\[\text{Table 1: Baseline calibration. A } \dagger \text{ (}) \text{ denotes a parameter that only applies to Model 1 (Model 2), unless specified.}\]

| Frisch Elasticity of Labor Supply | 1/\eta | 3 | Inflation Coefficient: MP Rule | \( \phi_x \) | 1.5 |
| Elasticity of Substitution between Goods | \theta | 6 | Output Coefficient: MP Rule | \( \phi_y \) | 0.1 |
| Rotemberg Adjustment Cost Coefficient | \varphi | 59.11 | Steady-State Technology | \( \bar{\varepsilon} \) | 1 |
| Steady-State Labor | \( \bar{n} \) | 0.33 | Technology Persistence\dagger | \( \rho_s \) | 0.9 |
| Capital Depreciation Rate\dagger | \( \delta \) | 0.025 | Technology Shock Standard Deviation\dagger | \( \sigma_\delta \) | 0.0025 |
| Cost Share of Capital\dagger | \( \alpha \) | 0.33 | Steady-State Discount Factor | \( \beta \) | 0.995 |
| Capital Adjustment Cost\dagger | \( \phi \) | 5.6 | Discount Factor Persistence | \( \rho_\beta \) | 0.8 |
| Steady-State inflation | \( \bar{r} \) | 1.006 | Discount Factor Standard Deviation | \( \sigma_\beta \) | 0.0025 |

Initially, we specify that \( z_t = \bar{\varepsilon} \) to simplify the presentation of the solution, and later we draw a comparison to Model 1 via nonlinear impulse response functions assuming that \( z_t \) follows the exogenous process in (4).

The likelihood of hitting the ZLB depends critically on the parameters of the stochastic processes. A determinate
In the policy sector, the steady-state gross inflation rate, $\bar{\pi}$, is set to 1.006, which implies an annual inflation rate target of 2.4 percent. That value equals the average growth rate of the U.S. PCE chain-type price index from 1983-2007. In our baseline calibration, the coefficients on inflation and output in the policy rule are set to 1.5 and 0.1, but we also consider other values.

We solve the model using the policy function iteration algorithm described in Richter et al. (2013), which is based on the theoretical work on monotone operators in Coleman (1991). This solution method discretizes the state space and uses time iteration to solve for the updated policy functions until the tolerance criterion is met. We use piecewise linear interpolation to approximate future variables that show up in expectations, since this approach more accurately captures the kink in the policy functions than continuous functions, and Gauss-Hermite quadrature to numerically integrate. These techniques capture the expectational effects of going to and returning to the ZLB. For a more formal description of the numerical algorithm and convergence, see appendix A.

The models are simulated using draws from the distributions for the discount factor and technology, which requires that these parameters are not too large for a given coefficient on the output gap in the policy rule. In the data, deviations of log real GDP from trend are 1.85 percent per quarter and deviations of the log difference in the PCE price index are 0.29 percent from 1983 to 2007. The equivalent values in our models are smaller than is observed, but they do not include many real world shocks and sources of persistence needed to match the data.
nology shocks. The state space is discretized to minimize extrapolation of the policy functions during the simulation. As an example, figure 2 shows the simulated distributions for Model 1 when the central bank targets the steady-state output gap. We simulate the model for 500,000 quarters to obtain a large sample of ZLB events. Although not reported above, we also find that the likelihood of a discount factor shock causing the nominal interest rate to fall to zero is lower when the central bank targets the potential output gap rather than the steady-state output gap.

Figure 2a shows the unconditional distributions of technology, the discount factor, and the nominal interest rate. The state space for technology lies within $\pm 2.5$ percent of its steady-state value, which is normalized to unity. The state space for the discount factor lies between $\pm 1.9$ percent of the steady state, which equals 0.995. Over these states, the quarterly net nominal interest rate is distributed over a range of 0 to 3.6 percent, with a large mass (5 percent of the simulated quarters) between 0 and 20 basis points and a steady-state rate of 1.1 percent.

Figure 2b shows the distribution of the discount factor and technology conditional on the ZLB binding. When technology is high enough relative to the steady state and the central bank follows a Taylor rule, the nominal interest rate hits its ZLB. Fernández-Villaverde et al. (2012) also find that high levels of technology are associated with low interest rates. Kiley (2003) uses U.S. data to show that periods of high labor productivity growth have been associated with relatively low inflation and argues that this result could be caused by the Fed’s policy rule as our models suggest.

4 MODEL 1: STATES OF THE ECONOMY, ECONOMIC DYNAMICS, AND THE ZLB

The New Keynesian model without capital, laid out in section 2.1, contains two state variables, the discount factor and technology. This section presents the complete solution to that model, key cross sections of that solution, impulse responses to technology shocks, and simulation statistics. We compare these results across alternative monetary policy rules. Each variable is shown in percent deviations from its steady state, except inflation and the interest rates, which are net percentages.

Figure 3 shows three-dimensional contour plots of the nominal interest rate and adjusted output and over the entire state space, which provides a complete picture of the model solution for these variables when the central bank sets $y_t^* = \bar{y}$. The shaded areas represent the states of the economy where the net nominal interest rate, $\tilde{r}$, equals zero. This region illustrates that the nominal interest rate only hits the ZLB when either technology or the discount factor are unusually high. A higher discount factor makes the household more patient, which reduces their demand across the entire state space and causes the ZLB to bind at a lower technology state. When the central bank aggressively targets the steady-state output gap ($y_t^* = \bar{y}$), a higher level of technology lowers inflation, which reduces demand when the ZLB binds. Looking at the highest discount factor shown in figure 3, the same unconventional response in output occurs, even when technology is below its steady state. Indeed, many ZLB studies consider that area of the state space. A higher technology state only increases output when technology is well-below steady state.

The contours in figure 3 are useful because they provide the solution for every possible combination of the shocks, but they also can be difficult to read. Thus, we focus on specific cross

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9In all of our results, a hat denotes percent deviation from the deterministic steady state (i.e., for some generic variable $x$ in levels, $\hat{x}_t = 100(x_t - \bar{x})/\bar{x}$) and a tilde denotes a net rate (i.e., for some gross rate $x$, $\tilde{x}_t = 100(x_t - 1)$).

10These bounds are chosen so that they encompass 99.999 percent of the probability mass of the technology and discount factor distributions and to minimize extrapolation of the policy functions in simulations of the model. We also specify a very dense discretized state space, so the location of the kink in the policy function is accurate.
sections of the state space. The solid (black) line in figure 3 shows the cross section where the technology state is held constant at its steady state ($\hat{z}_{-1} = 0$). Two-dimensional representations of that cross section are shown in figure 4. The shaded region indicates where the ZLB binds, which begins in states where the discount factor is 0.9 percent above its steady state. A high discount factor indicates that households have a strong desire to save. In that case, household demand is depressed, which reduces output, inflation, and the nominal interest rate. At the ZLB, the elevated discount factor is expected to mean revert, which puts upward pressure on expected inflation and the real interest rate. That higher real interest rate then makes current consumption more costly and causes even lower household demand in discount factor states where the ZLB binds.

The dashed (blue) line in figure 3 shows the cross section where the discount factor is held constant at 0.9 percent above its steady-state value ($\hat{\beta}_{-1} = 0.9$), which is the minimum value where the ZLB binds when technology equals its steady state. Two-dimensional representations of that cross section are shown in figure 5, which also includes results with alternative weights on the steady-state output gap ($\phi_y$). The entire shaded region indicates where the ZLB binds when $\hat{z}_{-1} = 0$ and $\phi_y = 0$. Larger values of $\phi_y$ cause the ZLB to first bind in slightly higher technology states, as the darker shaded regions show. The unconventional response of the economy to a positive technology shock is smaller as the value of $\phi_y$ declines. With $\phi_y = 0.05$ ($\phi_y = 0$), the response of output is positive in technology states up to 0.75 (1.4) percent above its steady state. Moreover, in the high technology states where the economy does contract, output and inflation are more stable with a lower $\phi_y$. As an example, when $\phi_y = 0$ output never falls below its pre-ZLB level ($\hat{y}^{adj} = -1.25$), even in the highest technology state. In contrast, when $\phi_y = 0.1$, output falls from $-1$ to $-3$ percent when the technology increases state from $\hat{z}_{-1} = 0$ to $\hat{z}_{-1} = 2.5$. These results confirm the finding in Braun and Körber (2011) that a shorter expected duration at the
ZLB can reverse the unconventional dynamics, since the expected duration of the ZLB increases in higher technology states. Their paper, however, sets $\phi_y = 0$ and does not consider alternative monetary policy specifications, which we demonstrate is also important.

To understand these results, we begin by examining the region of the state space where the ZLB does not bind. In low technology states, workers are less productive and firms’ per unit marginal cost of production is higher. Firms choose higher prices and have a low demand for labor. With less output available for consumption, the household works more to moderate the decline in consumption. The higher labor supply dominates the drop in labor demand so that the equilibrium level of labor is higher and the real wage is lower. The household also believes technology will slowly return to its steady state and, as a result, expects its future consumption to increase. Higher expected future consumption is reflected in an elevated real interest rate.

As we move to higher technology states before the ZLB binds, workers are more productive and firms choose lower prices and higher output. The household consumes more but also desires more leisure. In this part of the state space, the decline in the labor supply dominates the increase in labor demand so that labor hours are lower and the real wage is higher. Mean reversion in technology means that when technology is below steady state, the household expects lower future consumption growth and observes a lower real interest rate as technology reverts to its steady state.

A larger value of $\phi_y$ in technology states where the ZLB does not bind keeps output, labor, and the real wage rate closer to their steady states, but that additional stability comes at the expense of higher inflation and a higher nominal interest rate. The real interest rate is mostly unaffected.
Next consider technology states where the ZLB binds. In those states, higher technology continues to push down per unit production costs and firms react by lowering their prices. The additional decline in expected inflation combined with a zero nominal interest rate forces the real interest rate to rise. The household reduces its consumption to capitalize on the higher returns, which results in the paradox of thrift. Aggregate demand falls because everyone wants to save more at the higher real interest rate, but in the aggregate, it is not possible. Thus, the household reduces its consumption, which lowers output until actual and desired savings are equal. With less consumption, the household increases its labor supply. Firms respond to the reduction in demand by further lowering their prices and decreasing their output and labor demand. The drop in labor
demand dominates the increase in labor supply, so that both total hours and the real wage decline. This is an example of the paradox of toil [Eggertsson (2010)]. At the ZLB, everyone wants to work more, but the higher real interest rate lowers demand, which causes firms to reduce employment.

With less weight on the steady-state output gap, inflation is more stable in all of the technology states. Thus, the real interest rate rises less at the ZLB, which helps maintain household demand in high technology states. Higher labor demand raises equilibrium hours, which mitigates the decline in the real wage. In short, a tension exists at the ZLB between the supply-side effects of technology and the demand-side effects of the real interest rate. When the central bank responds less aggressively to the steady-state output gap when the ZLB does not bind (i.e., a lower $\phi_y$), the demand-side effects at the ZLB are weaker and both real and nominal variables are less volatile.

Figure 6 reports the impulse responses to a one-time 1 percent positive technology shock when the central bank targets the steady-state output gap under two alternative scenarios: (1) the steady-state scenario (solid line), where the ZLB does not bind is initialized at the stochastic steady state with $\beta$ equal to its deterministic steady state; and (2) the ZLB scenario (dashed line), where a sequence of discount factor shocks keep $\bar{\beta} = 0.9$. The horizontal dotted lines are the stochastic steady-state values of inflation and the (net) interest rates, which differ from the deterministic steady state due to the expectational effects of hitting the ZLB. In short, this figure compares the conventional impulse responses to a positive technology shock when the ZLB never binds to the responses based on a counterfactual where successive discount factor shocks keep the interest rate at zero. Intuitively, the series of discount factor shocks could represent a persistent reduction in consumer confidence, an ongoing global savings glut, or a decision by the Fed to hold the policy rate at zero. An advantage of looking at impulse responses over the policy functions is that they provide a clearer quantitative comparison between economic dynamics at and away from the ZLB.

The results in the steady-state scenario are standard and follow the intuition from the policy functions. A persistent technology shock lowers firms’ per unit marginal cost of production, increases output, and causes inflation and the nominal interest rate to fall. According to the Taylor rule, the nominal interest rate falls more than the inflation rate, so there is also a decline in the real interest rate, which increases consumption. A positive technology shock acts as a positive labor productivity shock, which decreases the equilibrium level of labor and raises the real wage rate.

In the ZLB scenario, a positive technology shock has unconventional effects on output and the real wage rate. At the stochastic steady state with $\bar{\beta}_{-1} = 0.9$, the higher discount factor imposes slight deflation. A positive technology shock then generates further deflation. With the nominal interest rate constrained at zero, the real interest rate sharply rises, which causes both output and labor to fall. We know from the policy functions in figure 5 that when $\phi_y$ is lower, the sign-reversal in output and the real wage ceases to occur, but a positive technology shock would increase output less than when the ZLB does not bind because the real interest rises at the ZLB regardless of $\phi_y$. In both scenarios, technology returns to its steady state about 20 quarters after the initial shock.

Figure 7 plots the same cross section of the state space that is shown dashed (blue) line in figure 3 across three alternative specifications of monetary policy: (1) the central bank does not respond to output ($\phi_y = 0$, solid line); (2) the central bank responds to the steady-state output gap ($y^*_t = \bar{y}$, $\phi_y = 0.1$, dashed line); and (3) the central bank responds to the potential output gap ($y^*_t = y^*_t$, $\phi_y = 0.1$, circle markers). The shaded region indicates where the ZLB binds, but the technology state where the ZLB initially binds depends on the monetary policy rule. When $y^*_t = \bar{y}$ ($y^*_t = y^*_t$), the ZLB first binds when the technology state is 0.1 (0.25) percent above its steady state. The most interesting difference between the policy rules with $y^*_t = \bar{y}$ and $y^*_t = y^*_t$ is that
higher technology states at the ZLB generate further increases in output and the real wage rate with a potential output gap as opposed to a sharp decline with the steady-state output gap.

In technology states below (above) steady state, the magnitude of the potential output gap is smaller (larger) than the steady-state output gap. This means that when the central bank targets potential output, it responds more (less) aggressively in low (high) technology states and therefore inflation is relatively more stable in all of the technology states. In high technology states, firms’ per unit marginal costs are low, which decreases inflation, increases the real interest rate, and reduces demand. The weaker reaction by the central bank when \( y^*_t = y^*_n \) helps boost demand. The higher demand at the ZLB means there is relatively less deflation, which mitigates the rise in the
real interest rate. A much lower increase in the real interest rate ultimately means that the demand-side effects are relatively weaker and output does not fall in high technology states. These results shed new light on the sources of the unconventional dynamics at the ZLB. While it is true that the expected duration of the ZLB (determined by the technology state) can reverse the unconventional dynamics at the ZLB, we contend that the specification of monetary policy plays a larger role.

In terms of the monetary policy rule, the Federal Reserve has favored a version of the Taylor rule that puts weight on both inflation and the output gap. We noted above that the specification of the output gap qualitatively affects how the economy behaves in high technology states. Next,
we examine how it affects the likelihood of hitting the ZLB using 500,000 quarter simulations of
the models. Our main result is that policymakers who more aggressively react to the steady-state
(potential) output gap will increase (decrease) the likelihood of hitting the ZLB.

<table>
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Table 2: Volatility implications of alternative weights on the output gap. Model 1: No capital, technology and discount factor shocks. $\phi_x = 1.50, \rho_z = 0.90, \sigma_z = 0.0025, \rho_\beta = 0.80$, and $\sigma_\beta = 0.0025$.

<table>
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<tr>
<th>Steady-State Output ($y_t^* = \bar{y}$)</th>
<th>Potential Output ($y_t^* = y_t^{n}$)</th>
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</thead>
<tbody>
<tr>
<td>$\phi_x$ % of quarters</td>
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<td>---------------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>1.500</td>
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<tr>
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Table 3: Volatility implications of alternative weights on the inflation gap. Model 1: No capital, technology and discount factor shocks. $\phi_y = 0.125, \rho_z = 0.90, \sigma_z = 0.0025, \rho_\beta = 0.80$, and $\sigma_\beta = 0.0025$.

Table 2 shows the effect of reducing the relative weight on output ($\phi_y$) while holding the weight
on inflation constant at $\phi_x = 1.5$. We begin with the original Taylor (1993) specification, $\phi_y = 0.125$, and reduce this coefficient by increments of 0.025. With the steady-state output gap ($y_t^* = \bar{y}$), the ZLB binds in 2.73 percent of the quarters in our simulation when $\phi_y = 0.125$. This value monotonically decreases with $\phi_y$ and equals 2.33 percent when $\phi_y = 0$. Decreasing the weight on the steady-state output gap raises the volatility of output but has no meaningful effect on the volatility of inflation. Overall, there is not much of a tradeoff between the volatility of output and inflation. The results are reversed when the central bank reacts to the potential output gap ($y_t^* = y_t^{n}$). Placing more weight on the potential output gap reduces the likelihood of hitting the ZLB and the standard deviations of output and inflation.11 These results are consistent with

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11If there was inertia in the monetary policy rule, then the frequency of ZLB events would significantly decline. For example, if the persistence in the Taylor rule equals 0.6, then the likelihood of the ZLB decreases to 0.03 percent of quarters given our baseline calibration. We chose to study the New Keynesian model without any persistence in the Taylor rule, because it removes a state variable and allows us to present the complete model solutions. While this assumption affects our quantitative results, none of our qualitative results rest on this assumption.
the main finding of Adam and Billi (2006) that it is optimal for the central bank to aggressively reduce the nominal interest rate in response to adverse shocks when it targets potential output. This reduces visits to the ZLB and welfare losses. We extend this analysis by drawing a comparison between the Models 1 and 2 in section 6 when only the discount factor is stochastic in Model 1.

Table 3 reports the results when we fix $\phi_y = 0.125$ and change the weight on the inflation gap ($\phi_\pi$). The frequency of ZLB events is more sensitive to changing $\phi_\pi$ than changing $\phi_y$. With $y_t^* = \bar{y}$, the likelihood of hitting the ZLB falls from 2.73 percent of quarters in the baseline case to 0.43 percent when $\phi_\pi = 3$. Moreover, the standard deviations of output and inflation fall as $\phi_\pi$ is increased. Increasing $\phi_\pi$ also reduces the likelihood of hitting the ZLB when $y_t^* = y_t^\pi$. In the baseline case, the ZLB binds in 1.56 percent of quarters and 0.43 percent of quarters when $\phi_\pi = 3$.

5 Model 2: States of the Economy and the ZLB

This section shows how the implications of the New Keynesian model change when capital accumulation is taken into account. The only way for the household to smooth consumption in Model 1 is by varying its labor supply. The advantage of including capital is that it gives the household another margin to smooth consumption. The results in this section are based on a model without technology shocks so that we can present the complete solution. Thus, this model contains two state variables—the discount factor and the capital stock, which is endogenous. Since there is no analytical solution for potential output in Model 2 and numerically the New Keynesian model with a ZLB constraint cannot be solved with flexible prices [Richter and Throckmorton (2013)], we only consider the case where the central bank targets the steady-state output gap in its policy rule.

Figure 8 shows the three-dimensional contour plots of the nominal interest rate, output, consumption, and investment over the entire state space. Presenting a complete picture of the model solution is particularly important in models with an endogenous state variable. The contour plots show these endogenous dynamics, which cross sections of the state space cannot capture. The curvature of the ZLB (shaded) region is due to the quadratic capital adjustment costs. When capital equals its steady state ($\hat{k}_{-1} = 0$), the ZLB binds when $\hat{\beta}_{-1} = 1.22$. As capital rises, the nominal interest rate equals zero at lower levels of the discount factor. In general, the patterns for consumption, inflation, and the nominal interest rate are similar to the model without capital. The household’s ability to invest in capital, however, causes consumption to be less volatile and generates stronger expectational effects than in Model 1.

Despite its drawbacks, two cross sections of the contour map in figure 8 are examined. The endogeneity of capital makes selecting particular cross sections in Model 2 more difficult than in Model 1. In Model 1, the discount factor and technology states are independent and, therefore, any one realization of the discount factor is just as likely regardless of the technology state. In Model 2, the capital and discount factor states are not independent so that the capital state is likely below (above) its steady state when the discount factor is also below (above) its steady state.

Figure 9 shows two cross sections from the contour map in figure 8. The solid (black) line is the cross section where the capital state is held fixed at its steady-state value ($\hat{k}_{-1} = 0$). The dashed (blue) line represents the cross section where capital increases along the diagonal of the state space ($k_{-1} = k_{\text{diag}}$). The darker (entire) shaded region indicates the area of the state space where the ZLB binds in the steady-state (diagonal) cross section. We begin by examining the behavior of the economy when the ZLB does not bind. Regardless of the capital state, a higher discount factor makes the household more patient, which increases their desire to invest in capital and postpone...
consumption. Lower consumption and higher output reduces inflation. To smooth its consumption across time, the household increases its labor supply. Moreover, with a higher capital stock, the marginal product of labor is higher, and labor demand also increases. Therefore, the equilibrium level of labor rises and the real wage rate falls as the discount factor increases. An increasing capital stock reduces its marginal product, causing the real rental rate to fall.

In the diagonal cross section where the capital state increases with the discount factor state ($\hat{k}_{-1} = \hat{k}_{\text{diag}}$), the marginal product of capital is lower in higher discount factor states, leading to a more rapid decline in the real rental rate. From the household’s perspective, this makes investment less attractive as a consumption smoothing channel. The household responds by decreasing con-

Figure 8: Model 2 ($y_t^* = \bar{y}$) decision rules as a function of the discount factor ($\hat{\beta}_{-1}$) and capital ($\hat{k}_{-1}$) states. Each variable is in percent deviations from its deterministic steady state, except the nominal interest rate, which is a net percentage. The shaded region indicates where the ZLB binds. The solid (black) and dashed (blue) lines correspond to cross sections of the state space where $\hat{k}_{-1} = 0$ and $\hat{k}_{-1} = \hat{k}_{\text{diag}}$. 
Figure 9: Model 2 ($y^*_t = \bar{y}$) decision rules as a function of the discount factor state ($\hat{\beta}_{-1}$). The solid line is the cross section of the state space where the capital state is fixed at its steady-state value ($\hat{k}_{-1} = 0$) and the dashed line is the diagonal cross section where the capital state changes with the discount factor state ($\hat{k}_{-1} = \hat{k}_{diag}$). Each variable is in percent deviations from its deterministic steady state, except inflation and the nominal interest rate, which are net percentages. The dark (entire) shaded region indicates where the ZLB binds when $\hat{k}_{-1} = 0$ ($\hat{k}_{-1} = \hat{k}_{diag}$).

umption and increasing labor supply less than in the cross section where the capital state is held fixed at its steady state ($\hat{k}_{-1} = 0$). This different behavior is apparent from the slopes of the real rental rate, investment, consumption, and labor policy functions.

In the diagonal (steady-state) cross section, the ZLB binds when the discount factor is more than 0.8 (1.2) percent above its steady state. The qualitative properties of the policy functions when the ZLB binds are similar across both cross sections. The mechanism that distorts the economy is similar to Model 1. As households become increasingly more patient in high discount factor states, consumption and inflation continue to fall. With the nominal interest rate pegged at zero, the real interest rate rises. When household demand falls, both consumption and investment decrease. Firms respond to the lower demand by further reducing their prices and sharply cutting their labor demand, which causes the equilibrium level of labor and the real wage rate to fall. Lower consumption and investment pushes down output. As output falls, the household further reduces its investment to smooth consumption. Thus, the paradoxes of toil and thrift both occur—despite the household wanting to work more to smooth consumption and save more to benefit from higher real interest rates, output contracts and both employment and investment fall. These results demonstrate that Model 2 faces the same unconventional dynamics as Model 1.

The Importance of Nonlinearities. We apply the policy function iteration algorithm to log-linearized versions of Model 1 and Model 2, where the only nonlinearity in the system of equations is the ZLB constraint. This produces a similar solution to the one in Nakov (2008), which also employs linear splines to better approximate the kink in the decision rules. We then compare the resulting linear decision rules to their fully nonlinear counterparts to demonstrate the importance of conducting our analysis with the nonlinear model. The benefit of this approach is that the solution method is identical for both specifications so that the only difference in the decision rules stems from whether or not the model is linearized.

![Figure 10: Model 1 ($\hat{y}_t = \bar{y}$) decision rules as a function of the technology state (left panel) and the discount factor state (right panel). The solid line (dashed line) corresponds to the decision rules based on the constrained nonlinear (linear) model. Each variable is in percent deviations from its deterministic steady state. The dark (entire) shaded region indicates where the ZLB binds in the fully nonlinear (linear) model.](12)

12The rental rate of capital falls at the ZLB, but the household expects that the future rental rate will increase since they believe the discount factor will return to its steady state. That result is consistent with a rising real interest rate.
Figure 10 compares cross sections of the linear and nonlinear decision rules for output in Model 1 when the central bank targets the steady-state output gap. The left (right) panel shows the decision rule as a function of the technology state (discount factor state). The linear decision rules are fairly accurate as a function of the technology states up until the ZLB binds (the shaded region). The wedge between the nonlinear and linear decision rules for output comes from the inability of monetary policy to compensate for growing price adjustment costs, which are different due to the linearization of the quadratic price adjustment cost function. The same is true of the linear decision rule as a function of the discount factor. If the bounds of the discount factor states were wider, then we would also see a growing wedge between the linear and nonlinear decision rules for output deeper into the ZLB region. In Model 1, the location of the kink in the linear and nonlinear decision rules is only slightly different. Overall the linear model provides a fairly good approximation of the nonlinear model without capital in most states, contrary to the finding in Fernández-Villaverde et al. (2012) who rely on continuous approximating functions. However, when we add capital in Model 2, the linear model is far less accurate due to the expectational effects of the ZLB.

Figure 11: Model 2 ($y^*_t = \bar{y}$) decision rules as a function of the discount factor state ($\hat{\beta}_{-1}$). The capital state changes with the discount factor state ($k_{-1} = \hat{k}_{\text{diag}}$). The solid line (dashed line) corresponds to the decision rules based on the constrained nonlinear (linear) model. Each variable is in percent deviations from its deterministic steady state, except inflation and the interest rates, which are net percentage. The dark (entire) shaded region indicates where the ZLB binds in the fully nonlinear (linear) model.

Figure 11 compares the linear and nonlinear decision rules in Model 2 as a function of the discount factor state. The ZLB first binds in the linear (nonlinear) model when technology is 1.4 (0.8) percent above its steady state. That difference has two important implications. First, it means that simulations of the linear model visit the ZLB far less frequently than the nonlinear model. In
fact, given our baseline calibration, the ZLB never binds in the linear model, while it binds in 1.15 percent of quarters in the nonlinear model. This is one reason why ZLB studies that linearize the equilibrium system rely on much larger shocks to generate ZLB events. Second, it means that the decision rules differ when the ZLB does not bind, since the expectational effects of visiting the ZLB are much weaker in the linear model. Thus, the linear model cannot accurately quantify the effects of discount factor and technology shocks even when the ZLB does not bind. The reason is because unlike Model 1, which only has one asset, Model 2 has two assets—capital and bonds. Arbitrage implies that the expected rates of return on investment and bonds are equal. Thus, the expected future rental rate of capital positively co-moves with the current real interest rate. It is interesting that in the nonlinear model, the real interest rate falls in discount factor states that are high, but not high enough for the ZLB to bind. In these states, the household places substantially more weight on the shocks that cause the ZLB to bind. Thus, the unconventional dynamics that occur at the ZLB cause the household to expect the rental rate of capital to fall and consumption growth to slow. Both of these effects cause the real interest rate to fall before the ZLB is hit. The linear model misses these important expectational effects. Overall, linearization drastically changes the predictions of Model 2, making it unsuitable for analyses of the ZLB.

6 Model 1 and Model 2 Comparisons

This section shows that capital accumulation qualitatively and quantitatively affects dynamics at the ZLB. To show that result, the impulse responses of the model without capital (Model 1) are compared to those responses from the model with capital (Model 2). We evaluate impulse response functions rather than comparing the models across a particular cross section of the state space, because specific assumptions about how the capital state in Model 2 co-moves with the exogenous state variables would need to be made. To conduct such an experiment, we include a stochastic process for technology with the same parameter values as Model 1 in Model 2. We also assume that the central bank targets the steady-state output gap \( y^*_\text{t} = \bar{y} \) and set \( \phi_y = 0.025 \) in both models. A small weight on \( \phi_y \) is necessary to make a direct comparison because Model 2 does not have a determinate solution for higher values of \( \phi_y \).

Figure 12 plots the impulse responses to one-time 1 percent positive technology shock in Model 1 (solid line) and Model 2 (dashed line) when the discount factor is held fixed at the minimum value that causes the ZLB to bind in each model (\( \hat{\beta}^{\text{Model 1}} = 0.89 \) and \( \hat{\beta}^{\text{Model 2}} = 1.05 \)). The horizontal dotted lines are the stochastic steady-state values of inflation and the (net) nominal interest rate. Those values are lower in Model 2 than in Model 1 because the discount factor is held at a higher value in Model 2. First note that the unconventional dynamics do not show up in Model 1, since the positive supply-side effects of higher technology dominate the negative demand-side effects of a higher real interest rate when \( \phi_y \) is small. As \( \phi_y \) increases, the relative strength of the demand-side effect increases, and therefore, output and labor hours both decline in response to a positive technology shock at the ZLB. The important takeaway from these results is that output declines in Model 2 even when \( \phi_y \) is small. Moreover, the magnitude of the response of output is four times larger in Model 2 than in Model 1. The responses of the real interest rate, inflation, and labor are qualitatively the same in both models, but are quantitatively larger in Model 2, even after controlling for differences in the stochastic-steady state values at the ZLB.

Next, we analyze the impact of capital accumulation on the volatility of output and inflation for a range of values for \( \phi_y \). Model 1 and Model 2 are simulated for 500,000 quarters under
Figure 12: Impulse responses to a 1 percent positive technology shock. The discount factor is constant and equal to the minimum value that ensures the ZLB always binds. In both cases, the central bank targets the steady-state output gap ($y^*_t = \bar{y}$). The horizontal dotted lines are the stochastic steady values of inflation and the (net) nominal interest rate. The other variables are given in percent deviations from their respective stochastic steady states.

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<th>Std. Dev. (% of mean)</th>
<th>ZLB Binds</th>
<th>Std. Dev. (% of mean)</th>
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Table 4: Volatility implications of alternative weights on the output gap. Comparison between Model 1 and Model 2. The only stochastic component in both models is discount factor shocks. $\phi_\pi = 1.50$, $\rho_\beta = 0.80$, and $\sigma_\beta = 0.0025$.

the assumption that the central bank targets the steady-state output gap. The stochastic process for technology is also removed from both models so that a broader range of values for $\phi_y$ can be examined. Table 4 shows the effect of reducing the relative weight on the output gap ($\phi_y$) while holding the weight on inflation ($\phi_\pi$) constant at 1.5. The value of $\phi_y$ is initially set slightly below the original Taylor (1993) specification, $\phi_y = 0.1$, and is reduced by increments of 0.025. Our results show that a larger value of $\phi_y$ in Model 1 decreases the likelihood of ZLB events. At first glance, this result appears to contradict our findings from Model 1 with the steady-state output gap that is reported in Table 2, column 2. In these results, however, there are no technology
shocks, which means the steady-state output gap and the potential output gap are equivalent. Thus, the qualitative result is consistent with the findings from Model 1 with the potential output gap that is presented in table 2, columns 5. Model 2, in contrast, generates the opposite result of Model 1 in two ways. First, a higher $\phi_y$ increases the likelihood ZLB events in Model 2. Second, both output and inflation volatility decline as $\phi_y$ increases in Model 1, whereas a tradeoff exists between lower output volatility and higher inflation volatility in Model 2. Those differences are particularly important given that many central banks have placed a great deal of emphasis on output stabilization since the end of the Great Recession.

7 Conclusion

This paper calculates global nonlinear solutions to standard New Keynesian models with and without capital. Our analysis studies discount factor shocks since they dominate the literature as a cause of ZLB events, but we also examine technology shocks because they are an important source of aggregate fluctuations in many dynamic models. We use these models to analyze why technology shocks at the ZLB may have unconventional effects, the likelihood of hitting the ZLB, and the tradeoffs faced by the central bank under a dual mandate. Three main findings emerge:

1. A positive technology shock can generate lower consumption, labor, and output—what we call unconventional dynamics—when the ZLB binds, but the specification of the output gap in the Taylor rule may reverse these effects.

2. When the central bank targets the steady-state output gap in the model with capital, a positive technology shock at the ZLB leads to more pronounced unconventional dynamics than in the model without capital.

3. The constrained linear model provides a decent approximation of the nonlinear model without capital, but meaningful differences exist between the solutions in the model with capital.

Despite the large volume of work on the ZLB, many important questions remain. For example, do the medium- to long-run benefits of returning to normal policy outweigh the short-run costs of a higher nominal interest rate? What are the benefits of forward guidance and quantitative easing in a dynamic model that accounts for expectational effects, and how do these policies change the effects of discount factor and technology shocks? Answering these questions and others will require careful treatment of expectations and is the subject of ongoing research.
REFERENCES


A Numerical Algorithm

A formal description of the numerical algorithm begins by writing the model compactly as

$$
E[f(v_{t+1}, w_{t+1}, v_t, w_t)|\Omega_t] = 0,
$$

where $f$ is vector-valued function that contains the equilibrium system, $v$ is a vector of exogenous variables, $w$ is a vector of endogenous variables, and $\Omega_t = \{M, P, z_t\}$ is the household’s information set in period $t$, which contains the structural model, $M$, its parameters, $P$, and the state vector, $z$. In model 1, $v = z = (\beta, z)$ and $w = (c, \pi, y, n, w, r)$. In model 2 with both stochastic processes, $v = (\beta, z)$, $z = (k, \beta, z)$, and $w = (c, \pi, y, n, w, r, k, i, r_k, q)$.

Policy function iteration approximates the vector of decision rules, $\Phi$, as a function of the state vector, $z$. The time-invariant decision rules for the exogenous model are

$$
\Phi(z_t) \approx \hat{\Phi}(z_t).
$$

We choose to iterate on $\Phi = (n, \pi)$ for Model 1 and $\Phi = (n, \pi, i)$ for Model 2 so that we can easily solve for future variables that enter the household’s expectations using $f$. Each continuous state variable in $z$ is discretized into $N^d$ points, where $d \in \{1, \ldots, D\}$ and $D$ is the dimension of the state space. The discretized state space is represented by a set of unique $D$-dimensional coordinates (nodes). In general, we set the bounds of continuous stochastic state variables to encompass 99.999 percent of the probability mass of the distribution. We specify 251 grid points for each continuous state variable and use 31 Gauss-Hermite weights for each continuous shock. These techniques minimize extrapolation and ensure that the location of the kink in the decision rules is accurate.

The following outline summarizes the policy function algorithm we employ for our models. Let $i \in \{0, \ldots, I\}$ index the iterations of the algorithm and $n \in \{1, \ldots, \Pi_{d=1}^D N^d\}$ index the nodes.

1. Obtain initial conjectures for the approximating functions on each node from the log-linear model without the ZLB imposed. The approximating functions are $\hat{c}_0$ and $\hat{n}_0$ for Model 1 and $\hat{\pi}_0$, and $\hat{i}_0$ for Model 2. We use \\texttt{gensys.m} to obtain these conjectures.

2. For $i \in \{1, \ldots, I\}$, implement the following steps:
   
   (a) On each node, solve for $\{y_t, r_t\}$, given $\hat{c}_{i-1}(z^n_t)$ and $\hat{\pi}_{i-1}(z^n_t)$ in Model 1 and given $\hat{n}_{i-1}(z^n_t)$, $\hat{\pi}_{i-1}(z^n_t)$, and $\hat{i}_{i-1}(z^n_t)$ in Model 2 with the ZLB imposed.
   
   (b) Linearly interpolate $\{c_{t+1}, \pi_{t+1}\}$ in Model 1 and $\{n_{t+1}, \pi_{t+1}, i_{t+1}\}$ in Model 2 given $\{z^n_{t+1}\}_{M_{t+1}}$. Each of the $M$ values $z^n_{t+1}$ are Gauss-Hermite quadrature nodes. We use Gauss-Hermite quadrature to numerically integrate, since it is accurate for normally distributed shocks. We use piecewise linear interpolation to approximate future variables, since this approach more accurately captures the kink in the decision rules than continuous functions such as cubic splines or Chebyshev polynomials.  

\textsuperscript{13}Aruoba and Schorfheide (2013) use a linear combination of two Chebyshev polynomials—one that captures the dynamics when the ZLB binds and one that captures the dynamics when the Taylor principle holds. While this approach is more accurate than using one Chebyshev polynomial, there is no guarantee that it will accurately locate the kink. Moreover, Chebyshev polynomials can lead to large approximation errors due to extrapolation. With linear interpolation, a dense state space will lead to more predictable extrapolation and more accurately locate the kink.
(c) We use the nonlinear solver, csolve.m, to minimize the Euler equation errors. On each node, numerically integrate to approximate the expectation operators,

\[ E[f(x_{t+1}^m, x_t^n)|\Omega_t] \approx \frac{1}{\pi} \sum_{m=1}^{M} f(x_{t+1}^m, x_t^n) \phi(\varepsilon_{t+1}^m), \]

where \( x \equiv (v, w) \) and \( \phi \) are the respective Gauss-Hermite weights. The superscripts on \( x \) indicate which realizations of the state variables are used to compute expectations. The nonlinear solver searches for \( \hat{c}_i(z^n_t) \) and \( \hat{\pi}_i(z^n_t) \) in Model 1 and \( \hat{n}_i(z^n_t) \), \( \hat{\pi}_i(z^n_t) \), and \( \hat{i}_i(z^n_t) \) in Model 2 so that the Euler equation errors are less than \( 1^{-4} \) on each node.

3. Define \( \maxdist_i \equiv \max\{|\hat{c}_i - \hat{c}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|\} \) in Model 1 and \( \maxdist_{i} \equiv \max\{|\hat{n}_i - \hat{n}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|, |\hat{i}_i - \hat{i}_{i-1}|\} \) in Model 2. Repeat the steps in item 2 until one of the following conditions is satisfied:

- If for all \( n \), \( \maxdist_i < 1^{-13} \) for 10 consecutive iterations, then the algorithm converged to the unique bounded MSV solution. In Model 1, since the state is composed of only exogenous variables, the solution is bounded so long as the decisions rules are positive and finite. In Model 2, simulations of the model must not be explosive.

- Otherwise, we say the algorithm is non-convergent for one of the following reasons:
  - \( i = I = 500,000 \) (Algorithm times out)
  - For all \( n \) and any \( i \), \( \hat{\pi}_i < \frac{1}{2} \), or for any \( n \), \( \hat{c}_i < 0 \) in Model 1 or \( \hat{n}_i < 0 \) in Model 2 (Approximating functions drift)
  - Define \( \text{dir}_i = \maxdist_i - \maxdist_{i-1} \). For all \( n \), \( \text{dir}_i \geq 0 \) and \( \text{dir}_i - \text{dir}_{i-1} \geq 0 \) for 50 consecutive iterations (Algorithm diverges)

To provide evidence that the solution is unique, we randomly perturb the converged decision rules and check that the algorithm converges back to the same solution.