
ARTHUR D. LITTLE, INC., ACORN PARK, CAMBRIDGE, MA 02140.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A. P. HILLMAN, G. L. ALEXANDERSON, L. F. KLOSINSKI

The following results of the thirty-sixth William Lowell Putnam Competition, held on December 6, 1975, have been determined in accordance with the regulations governing the competition. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, was awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Christopher L. Henley, Frank M. Liang, and James B. Shearer; a prize of one hundred dollars was awarded to each of these students.

The second prize, four hundred dollars, was awarded to the Department of Mathematics of the University of Chicago, Chicago, Illinois. The members of the team were Franklin T. Adams, Peter L. Dordal, and Andrew M. McLennan; each was awarded a prize of seventy-five dollars.

The third prize, three hundred dollars, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were David J. Anick, Sheldon H. Katz, and Richard E. Stone; each was awarded a prize of fifty dollars.

The fourth prize, two hundred dollars, was awarded to the Department of Mathematics of Princeton University, Princeton, New Jersey. The members of the team were Robert L. Anderson, Eric S. Lander, and Karl C. Rubin; each was awarded a prize of fifty dollars.

The fifth prize, one hundred dollars, was awarded to the Department of Mathematics of Harvard
University, Cambridge, Massachusetts. The members of the team were Sholom Feldblum, Thomas G. Goodwillie, and Alan S. Grenadie; each was awarded a prize of fifty dollars.

The five highest ranking individual contestants, in alphabetical order, were Franklin T. Adams, University of Chicago; David J. Anick, Massachusetts Institute of Technology; Ernest S. Davis, Massachusetts Institute of Technology; Thomas G. Goodwillie, Harvard University; and Christopher L. Henley, California Institute of Technology. Each of these students has been designated as a Putnam Fellow by the Mathematical Association of America and was awarded a prize of two hundred and fifty dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were H. Turner Laquer, University of New Mexico; Joseph E. Quinsey, University of Waterloo; Karl C. Rubin, Princeton University; James B. Shearer, California Institute of Technology; and Brian C. White, Yale University. Each of these students was awarded a prize of one hundred dollars.

The following teams, named in alphabetical order, won honorable mention: University of British Columbia with team members Paul A. Hildebrand, Allan D. Jepson, and Arnold D. Satanove; Carleton College with team members Max L. Benson, Albert T. Borchers, and Andrew A. Rich; University of Illinois, Urbana, with team members Bruce E. Hajek, Allen Moy, and Daniel D. Sleator; Michigan State University with team members Mark P. Merriman, Karl W. Pettis, and Ian H. Redmount; and University of Waterloo with team members Richard P. Anstee, Marks S. Brader, Gregory J. Fee.

Honorable mention was given to the following thirty individuals, named in alphabetical order: J. Eric Brosius, Pennsylvania State University; F. Michael Christ, Harvey Mudd College; Roy E. DeMee, Massachusetts Institute of Technology; Andrew Z. Fire, University of California, Berkeley; James A. Gardner, University of Waterloo; Ian P. Goulden, University of Waterloo; Philip I. Harrington, Washington University; Karl W. Heuer, Concordia College; David C. Hobby, University of Washington; Sheldon H. Katz, Massachusetts Institute of Technology; Donald T. Kersey, McMaster University; Nathaniel S. Kuhn, Harvard University; Frank J. Lhots, Wayne State University; Alan D. Listoe, University of Saskatchewan; Russell D. Lyons, Case Western Reserve University; Andrew M. McLennan, University of Chicago; Ross E. Millikan, University of California, Berkeley; Stephen W. Modzelewski, Harvard University; David R. Morrison, Princeton University; Allen Moy, University of Illinois, Urbana; N. Christopher Phillips, University of California, Berkeley; Nicholas S. Robins, University of California, Berkeley; Adam N. Rosenberg, Princeton University; David J. Rusin, Princeton University; Mark W. Saalitiek, University of Victoria; Roger S. Schalley, Princeton University; Matthew R. Smith, University of Waterloo; Daniel J. Velleman, Dartmouth College; Paul A. Vojta, University of Minnesota, Minneapolis; Norman J. Wildberger, University of Toronto.

The other individuals who achieved ranks among the top one hundred, in alphabetical order of their schools, were: Bowdoin College, Richard M. Crew; Brown University, Joseph H. Silberman; University of British Columbia, Allan D. Jepson, Arnold D. Satanove; California Institute of Technology, Robert W. Cox, Frank M. Liang, Stephen R. Roe, Douglas B. Tyler; University of California (Berkeley), Steven T. Tschantz; University of California (Davis), Stephen R. Peck; Carleton College, Albert T. Borchers, Andrew A. Rich; University of Chicago, Robert M. Beals, Peter L. Dordal, Ngaiming L. Mok; Drexel University, Dennis M. DeTurck; Georgia Institute of Technology, Michael E. Hoffman; Grinnell College, Dale R. Worley; Harvard University, Richard I. Anders, Robert F. Coleman, Peter G. Doyle, Sholom Feldblum, Tony G. Horowitz, Douglas W. Oman, David R. Richman, Vladislav G. Rutenburg, James P. Sethna; Holy Cross College, Gary B. Page; Indiana University, Philip H. Dyboig, Arvind N. Srivastava; Iowa State University, David L. Gordon; University of Maryland (College Park), Robert B. Bundy; Massachusetts Institute of Technology, Dean G. Sturtevant; McGill University, Gerald M. Cohen; Michigan State University, Mark P. Merriman, Karl W. Pettis; University of North Carolina, Wesley H. Presler; Oberlin College, Spencer W. Thomas; Pomona College, Eric V. Level; Princeton University, Michael J. Barall, Alan S. Geller,
Eric S. Lander, Jeffrey N. Rothman; Purdue University, Max C. Ng; Rice University, C. Robin Graham, John W. Myre; San Diego State University, John O. Lamping; Stanford University, Nick E. Baxter, John P. Dawson, Bruce A. Fast, Scott E. Kim; Union College, James B. Saxe; United States Naval Academy, Steven J. Raher; Wabash College, Thomas M. Sellke; Washington University, Tim J. Steiger, Philip N. Strenski; University of Waterloo, Mark S. Brader, Gregory J. Fee, Peter F. Schneider, Douglas R. Stinson; University of Windsor, Bradley J. Lucier; Worcester Polytechnic Institute, John A. Major; Yale University, David V. Feldman.

There were 2203 individual contestants from 355 colleges and universities in Canada and the United States in the competition of December 6, 1975. Teams were entered by 285 institutions.

The Questions Committee, consisting of G. D. Chakerian, J. D. E. Konhauser, and J. I. Richards (Chairman), prepared the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS, PART A

A–1. Supposing that an integer \( n \) is the sum of two triangular numbers,
\[
\frac{a^2 + a}{2} + \frac{b^2 + b}{2} = n,
\]
write \( 4n + 1 \) as the sum of two squares, \( 4n + 1 = x^2 + y^2 \), and show how \( x \) and \( y \) can be expressed in terms of \( a \) and \( b \).

Show that, conversely, if \( 4n + 1 = x^2 + y^2 \), then \( n \) is the sum of two triangular numbers.
[Of course, \( a, b, x, y \) are understood to be integers.]

A–2. For which ordered pairs of real numbers \( b, c \) do both roots of the quadratic equation
\[
z^2 + bz + c = 0
\]
lie inside the unit disk \( |z| < 1 \) in the complex plane?

Draw a reasonably accurate picture (i.e., "graph") of the region in the real \( bc \)-plane for which the above condition holds. Identify precisely the boundary curves of this region.

A–3. Let \( a, b, c \) be constants with \( 0 < a < b < c \). At what points of the set
\[
\{x^a + y^b + z^c = 1, \ x \geq 0, \ y \geq 0, \ z \geq 0\}
\]
in three-dimensional space \( R^3 \) does the function \( f(x, y, z) = x^a + y^b + z^c \) assume its maximum and minimum values?

A–4. Let \( n = 2m \), where \( m \) is an odd integer greater than 1. Let \( \theta = e^{i\pi/n} \). Express \( (1 - \theta)^{-1} \) explicitly as a polynomial in \( \theta \),
\[
a_0 + a_1 \theta + a_2 \theta^2 + \cdots + a_n \theta^n,
\]
with integer coefficients \( a_i \).

[Note that \( \theta \) is a primitive \( n \)-th root of unity, and thus it satisfies all of the identities which hold for such roots.]

A–5. On some interval \( I \) of the real line, let \( y_1(x) \) and \( y_2(x) \) be linearly independent solutions of the differential equation
\[
y'' = f(x)y,
\]
where \( f(x) \) is a continuous real-valued function. Suppose that \( y_1(x) > 0 \) and \( y_2(x) > 0 \) on \( I \). Show that there exists a positive constant \( c \) such that, on \( I \), the function
\[
z(x) = c \sqrt{y_1(x)y_2(x)}
\]
satisfies the equation
\[
z'' + \frac{1}{z^3} = f(x)z.
\]
State clearly the manner in which $c$ depends on $y_1(x)$ and $y_2(x)$.

A–6. Let $P_1, P_2, P_3$ be the vertices of an acute-angled triangle situated in three-dimensional space. Show that it is always possible to locate two additional points $P_4$ and $P_5$ in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

In writing your answer, state clearly the locations at which you place the points $P_4$ and $P_5$.

PROBLEMS, PART B

B–1. In the additive group of ordered pairs of integers $(m, n)$ [with addition defined componentwise: $\begin{pmatrix} m, n \end{pmatrix} + \begin{pmatrix} m', n' \end{pmatrix} = \begin{pmatrix} m + m', n + n' \end{pmatrix}$] consider the subgroup $H$ generated by the three elements 

\[(3, 8), \quad (4, -1), \quad (5, 4)\]

Then $H$ has another set of generators of the form 

\[(1, b), \quad (0, a)\]

for some integers $a, b$ with $a > 0$. Find $a$.

[Elements $g_1, \ldots, g_6$ are said to generate a subgroup $H$ if (i) each $g_i \in H$, and (ii) every $h \in H$ can be written as a sum $h = n_1 g_1 + \cdots + n_6 g_6$, where the $n_i$ are integers (and where, for example, $3g_1 - 2g_6$ means $g_1 + g_6 + g_6 - g_6$).]

B–2. In three-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$, respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

B–3. Let $s_k(a_1, \ldots, a_n)$ denote the $k$-th elementary symmetric function of $a_1, \ldots, a_n$. With $k$ held fixed, find the supremum (or least upper bound) $M_k$ of

\[s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k\]

for arbitrary $n \geq k$ and arbitrary $n$-tuples $a_1, \ldots, a_n$ of positive real numbers.

[The symmetric function $s_k(a_1, \ldots, a_n)$ is the sum of all $k$-fold products of the variables $a_1, \ldots, a_n$. Thus, for example:

\[s_1(a_1, \ldots, a_n) = a_1 + a_2 + \cdots + a_n;\]

\[s_2(a_1, a_2, a_3, a_4) = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4.\]

It should be remarked that the supremum $M_k$ is never attained; it is approached arbitrarily closely when, for fixed $k$, the number $n$ of variables increases without bound, and the values $a_i > 0$ are suitably chosen.]

B–4. Does there exist a subset $B$ of the unit circle $x^2 + y^2 = 1$ such that (i) $B$ is topologically closed, and (ii) $B$ contains exactly one point from each pair of diametrically opposite points on the circle?

[A set $B$ is topologically closed if it contains the limit of every convergent sequence of points in $B$.]

B–5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf'_n(x)$ for $n = 0, 1, 2, \ldots$. Show that

\[\sum_{n=0}^{\infty} f_n(1)/n! = e^x.\]

B–6. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n$, then

(a) $n(n+1)^{n-1} < s_n < n + s_n$ for $n > 1$, and
(b) $(n-1)n^{-1(n-1)} < s_n - s_{n-1}$ for $n > 2$.

SOLUTIONS

In the 12-tuples $(n_0, n_1, n_2, \ldots, n_{12})$ following each problem number below, $n_i$ for $10 \leq i \leq 0$ is the number of students, among the top 212 contestants, who received $i$ points for the problem and $n_{-1}$ is the number not submitting solutions.

A–1. 

\[152, 7, 6, 18, 3, 7, 0, 0, 0, 0, 1, 18, \ldots\]
Let \( n = [(a^2 + a)/2] + [(b^2 + b)/2] \), with \( a \) and \( b \) integers. Then
\[
4n + 1 = 2a^2 + 2a + 2b^2 + 2b + 1 = (a + b + 1)^2 + (a - b)^2.
\]
Conversely, let \( 4n + 1 = x^2 + y^2 \), with \( x \) and \( y \) integers. Then exactly one of \( x \) and \( y \) is odd and so \( a = (x + y - 1)/2 \) and \( b = (x - y - 1)/2 \) are integers. One easily verifies that
\[
[(a^2 + a)/2] + [(b^2 + b)/2] = (x^2 + y^2 - 1)/4 = n.
\]

A–2.
\[
(100, 6, 2, 0, 0, 5, 10, 3, 1, 27, 43, 15)
\]

The desired region is the inside of the triangle with vertices \((0, -1), (2, 1), (-2, 1)\). The boundary segments lie on the lines
\[
L_1: c = 1, L_2: c - b + 1 = 0, L_3: c + b + 1 = 0.
\]

To see this, we let \( f(z) = z^2 + bz + c \) and denote its zeros by \( r \) and \( s \). Then \(-b = r + s\) and \( c = rs\). Also
\[
(r + 1)(s + 1) = rs + r + s + 1 = c - b + 1 = f(-1),
\]
\[
(r - 1)(s - 1) = rs - r - s + 1 = c + b + 1 = f(1).
\]

On or below \( L_2 \), at least one zero is real and not greater than \(-1\); this follows either from \((r + 1)\cdot (s + 1) \leq 0\) or from \( f(-1) \leq 0\) and the fact that the graph of \( y = f(x) \), for \( x \) real, is an upward opening parabola. Similarly, on or below \( L_3 \), one zero is real and at least \(1\). On or above \( L_1 \), at least one zero has absolute value greater than or equal to \(1\). Hence the desired points \((b, c)\) must be inside the described triangle.

Conversely, if \((b, c)\) is inside the triangle, \(|c| < 1\) and so \(|r| < 1\) or \(|s| < 1\) or both. If the zeros are complex, they are conjugates and \(|r| = |s|\); then \(|r| = |s| < 1\) follows from \(|c| < 1\). If the zeros are real, \(|c| < 1\) implies that at least one zero is in \((-1, 1)\). Then \((r + 1)(s + 1) = f(-1) > 0\) and \((r - 1)(s - 1) = f(1) > 0\) imply that the other zero is also in \((-1, 1)\).

For full credit, the region had to be depicted.

A–3.
\[
(10, 6, 2, 2, 0, 8, 6, 2, 7, 23, 29, 112)
\]

Let \( h(x) = x^a - x^b \) and \( k(z) = z^c - z^b \). The desired points also give the maximum and minimum of the function
\[
g(x, z) = (x^a + y^b + z^c) - (x^b + y^b + z^b) = h(x) + k(z)
\]
on the domain obtained by projection of the solid domain on the \(xz\)-plane. For all points under consideration, both \(x\) and \(z\) are in \([0, 1]\). Examining its derivative, one sees that \( h(x) \) increases from 0 at \( x = 0 \) to a maximum at \( x_0 = (a/b)^{1/(b-a)} \) and then decreases to 0 at \( x = 1 \). (This uses the hypothesis \(0 < a < b\).) Similarly, \( k(z) \) decreases from 0 at \( z = 0 \) to a minimum at \( z_0 = (b/c)^{1/(c-b)} \) and then increases to 0 at \( z = 1 \). Since \((1, z_0)\) and \((x_0, 1)\) are not in the domain of \(g(x, z)\), the function \( f \) achieves its maximum only at \((x, y, z) = (x_0, [1 - x_0^{1/b}], 0)\) and achieves its minimum only at \((0, [1 - z_0^{1/b}], z_0)\).

A–4.
\[
(23, 3, 1, 0, 0, 2, 0, 0, 0, 0, 74, 109)
\]

Let \( n = 4k + 2 \) with \( k > 0 \). Then
\[
0 = \theta^n - 1 = \theta^{4k+2} - 1 = (\theta^{2k+1} - 1)(\theta + 1),
\]
\[
0 = (\theta^{2k+1} - 1)(\theta + 1)(\theta^2 - \theta^{2k-1} + \theta^{2k-2} - \cdots - \theta + 1)
\]
Since \( \theta \) is a primitive \(n\)th root of unity with \( n > 2k + 1 \) and \( n > 2 \),
\[
(\theta^{2k+1} - 1)(\theta + 1) \neq 0.
\]
Hence

\[\theta^{2k} - \theta^{2k-1} + \theta^{2k-2} - \cdots + \theta^2 - \theta + 1 = 0,\]
\[1 = \theta - \theta^3 + \theta^5 - \cdots - \theta^{2k} = (1 - \theta)(\theta + \theta^2 + \cdots + \theta^{2k-1}),\]
\[(1 - \theta)^{-1} = \theta + \theta^3 + \cdots + \theta^{2k-1} \text{ [where } 2k - 1 = (n - 4)/2].\]

Another solution is \((1 - \theta)^{-1} = 1 + \theta^2 + \theta^4 + \cdots + \theta^{2k}\) as one sees from (A).

A–5. 

\((22, 13, 2, 2, 3, 3, 2, 2, 2, 21, 137)\)

The answer for \(c\) is \(\sqrt{2/w}\), where \(w\) is the wronskian \(y_1 y'_2 - y_2 y'_1\) (and will be seen below to be constant).

Let \(c^2 = 2k\). Then \(z^2/2 = k y_1 y_2\). Differentiating twice, one has

\[zz' = k(y_1 y'_2 + y_2 y'_1), \quad zz'' + (z')^2 = k(y_1 y'_2 + y_2 y'_1 + 2y'_1 y'_2).\]

Since \(y'_1 = f y_1\) and \(y'_2 = f y_2\), this implies

\[zz'' + (z')^2 = 2k(f y_1 y_2 + y'_1 y'_2) = f(2k y_1 y_2) + 2k y'_1 y'_2 = f z^2 + 2k y'_1 y'_2.\]

Now

\[z^2 z'' + (z')^2 = f z^4 + 2k y'_1 y'_2,\]
\[z^2 z'' + k(y_1 y'_2 + y_2 y'_1)^2 = f z^4 + 4k y'_1 y'_2,\]
\[z^2 z'' + k(y_1 y'_2 + y_2 y'_1)^2 = f z^4,\]
\[z^2 z'' - f z^4 = - k^2(y_1 y'_2 - y_2 y'_1) = - k^2 w^2 = - c^4 w^2/4.\]

Since \(w' = (y_1 y'_2 - y_2 y'_1) = y_1 y'_2 - y_2 y'_1 = y_1 (f y_2) - y_2 (f y_1) = 0\), \(w\) is a constant. Solving \(c^4 w^2/4 = 1\) for \(c\) gives \(c = \sqrt{2/w}\); for this \(c\), (1) implies \(z'' - f z = - z^{-3}\) or \(z'' + z^{-3} = f z\).

A–6. 

\((3, 3, 5, 6, 3, 6, 6, 6, 26, 8, 29, 26, 91)\)

Let \(\lambda\) denote the line through the desired points \(P_1\) and \(P_2\). Let \(\pi\) be the plane of \(P_1\), \(P_2\), and \(P_3\) and let \(H\) be the intersection of \(\lambda\) with \(\pi\).

Let \(u_0\) be the vector \(HP_0\) and \(|u_0|\) be its magnitude. We wish to have the dot product

\[(1) \quad d = P_0 P_1 \cdot P_1 P_2 = (u_0 - u_h) \cdot (v_i - v_k) = u_h \cdot v_i - u_h \cdot v_k - v_i \cdot u_k + v_i \cdot v_k\]

zero for all choices of \(h, k, i, j\) as distinct indices in \(\{1, 2, 3, 4, 5\}\).

Since \(\lambda\) is to be perpendicular to \(\pi\), we must have

\[(2) \quad v_i \cdot v_k = 0 \quad \text{for } h \in \{4, 5\} \text{ and } i \in \{1, 2, 3\}.\]

If \(h, k \in \{4, 5\}\) and \(i, j \in \{1, 2, 3\}\), (2) implies that the dot product \(d\) of (1) is zero. If \(h \in \{4, 5\}\) and \(k, i, j \in \{1, 2, 3\}\), (2) implies that the \(d\) of (1) becomes

\[(3) \quad d = v_h \cdot v_j - v_k \cdot v_i = v_h \cdot (v_j - v_i) = HP_0 \cdot P_1 P_2.\]

Clearly the \(d\) of (3) are zero simultaneously if and only if \(H\) is the orthocenter (i.e., intersection of altitudes) of \(\Delta P_1 P_2 P_3\). With this choice of \(H\), the vanishing of the \(d\) of (3) implies

\[(4) \quad v_2 \cdot v_3 = v_1 \cdot v_3 = v_1 \cdot v_2.\]

Now let \(h, i \in \{4, 5\}\) and \(k, j \in \{1, 2, 3\}\). Then (2) implies

\[(5) \quad d = v_h \cdot v_j + v_k \cdot v_i.\]

Assuming (4), one sees that all the \(d\)'s of (5) will be zero if \(v_4 \cdot v_5 = - v_1 \cdot v_2\). The hypothesis that
\( \Delta P_1 P_2 P_3 \) is acute-angled tells us that \( H \) is inside the triangle. Then at least one (actually, all) of the angles \( \alpha P_1 H P_2, \alpha P_2 H P_3, \alpha P_1 H P_3 \) must be obtuse and so the equal dot products of (4) must be negative. Hence \( u_1 \cdot v_3 \) must be positive; this means that \( P_2 \) and \( P_3 \) must be on the same half-line of \( \lambda \) determined by \( H \).

Now the location of \( P_1 \) and \( P_2 \) can be given. Let \( H \) be the orthocenter of \( \Delta P_1 P_2 P_3 \) and \( \mu \) be either half-line perpendicular to plane \( P_1 P_2 P_3 \) at \( H \). Then \( P_1 \) may be any point on \( \mu \) such that \( |v_3| \) is neither zero nor \(( -v_1 \cdot v_2)/|v_2| \) and \( P_3 \) must be the unique point on \( \mu \) with \( |v_3| = -v_1 \cdot v_2/|v_2| \). Then each \( d \) of (1) is zero and no three of the \( P_i \) are collinear.

**B-1.**

(12, 6, 96, 6, 6, 46, 10, 3, 7, 5, 11, 4)

The answer is \( a = 7 \). Also one must have \( b = 5(\text{mod } 7) \).

**Proof:** The subgroup \( H \) must contain \( 4(3, 8) - 3(4, -1) = (0, 35), 4(5, 4) - 5(4, -1) = (0, 21) \), and then \( 2(0, 21) - (0, 35) = (0, 7) \). Now \( (0, 7) \) and \( (1, b) \) will generate \( H \) iff \( (1, b) \) is in \( H \) and there exist integers \( u, v, \) and \( w \) such that

\[
(3, 8) = 3(1, b) + u(0, 7), \quad (4, -1) = 4(1, b) + v(0, 7), \quad (5, 4) = 5(1, b) + w(0, 7).
\]

These hold iff \( 8 = 3b + 7u, -1 = 4b + 7v, \) and \( 4 = 5b + 7w \). With \( b = 5 + 7k, \) \( k \) any integer, the desired coefficients \( u, v, \) and \( w \) exist in the form \( u = -1 - 3k, v = -3 - 4k, \) \( w = -3 - 5k \). It now suffices to let \( k = 0 \) and to note that \( (1, 5) = (4, -1) - (3, 8) + 2(0, 7) \) is in \( H \).

**B-2.**

(79, 41, 4, 4, 1, 0, 1, 0, 3, 10, 31, 38)

Let \( \Sigma d = d \) and let \( S \) be a sphere of radius \( r > d/2 \). The area of \( S \) contained in slab \( S_t \) is at most \( 2\pi d \). It follows that the area of \( S \) contained in the union of the slabs \( S_t \) is at most \( 2\pi d < 4\pi r \) (area of \( S \)). Hence there are points of \( S \) that are not in any of the slabs.

The problem may also be done using volumes of intersection of the slabs with an appropriately chosen sphere.

**B-3.**

(43, 3, 6, 3, 1, 5, 2, 23, 18, 3, 24, 81)

In the expansion of \( s_t = (a_1 + a_2 + \cdots + a_6)^t \), every term of \( s_t \) appears with \( k! \) as coefficient and the other coefficients are nonnegative. Hence \( s_t/s_t^k \leq 1/k! \).

If we let each \( a_i = 1 \),

\[
\frac{s_t}{s_t^k} = \binom{n}{k}/n^k = \frac{n(n-1)\cdots(n-k+1)}{k!n^k} = \frac{1}{k!} \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right),
\]

which approaches \( 1/k! \) as \( k \) is held fixed and \( n \) goes to infinity. These facts show that the supremum \( M_t \) is \( 1/k! \).

**B-4.**

(73, 46, 15, 5, 1, 5, 2, 14, 4, 0, 32, 15)

No. Since the mapping with \((x, y) \to (-x, -y)\) is a homeomorphism of the unit circle on itself, the complement \(-B \) of such a subset \( B \) would also be closed. Thus the existence of such a \( B \) would make \( C \) the union \(-B \cup B \) of disjoint nonempty closed subsets; this would contradict the fact that \( C \) is connected.

**B-5.**

(14, 11, 17, 7, 2, 0, 4, 0, 1, 1, 27, 128)

Since \( f_n(x) = \Sigma_{k=0}^{n} x^k/k! \), one easily shows by mathematical induction that \( f_n(x) = \Sigma_{k=0}^{n} x^k/k! \). Then, since all terms are positive, one has

\[
\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{k^n}{k!n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k^n}{n!} = \sum_{k=0}^{\infty} \frac{e^k}{k!} = e^e.
\]
Both parts are done easily using the Inequality on the Means. For (a), one has
\[
\frac{n + s_n}{n} = \frac{(1 + 1) + (1 + \frac{1}{2}) + \cdots + (1 + \frac{1}{n})}{n} > \sqrt[n]{(1 + 1)(1 + \frac{1}{2})\cdots(1 + \frac{1}{n})} = \sqrt[n]{2 \cdot \frac{3}{2} \cdots \frac{n + 1}{n}} = (n + 1)^{1/n}
\]
and so \( n + s_n > n(n + 1)^{1/n} \).

For (b), one has
\[
\frac{n - s_n}{n - 1} = \frac{(1 - \frac{1}{2}) + (1 - \frac{1}{3}) + \cdots + (1 - \frac{1}{n})}{n - 1} > \sqrt[n]{(1 - \frac{1}{2})(1 - \frac{1}{3})\cdots(1 - \frac{1}{n})} = \sqrt[n]{\frac{1}{2} \cdot \frac{2}{3} \cdots (n - 1)/n} = n^{-1/(n-1)}
\]
and so \( n - s_n > (n - 1)n^{-1/(n-1)} \).

--

Vivienne Mayes

"No bubble is so iridescent or floats longer than that blown by the successful teacher."

William Osler

LEE LORCH AT FISK: A TRIBUTE

Vivienne Mayes

At the first Convocation held at Fisk University in September, 1950, Lee Lorch was introduced as the new chairman of the Department of Mathematics. Such a perfunctory ritual would have utterly failed to register in my student memory had not the president of Fisk, the late Charles S. Johnson, accompanied the introduction with an unusual story. He told how Dr. Lorch had lost his professorship at Penn State University because he had sublet his apartment to a Black family. This story aroused admiration in all of the students at the service. We were Black. And this story of a white sacrificing for a Black family in 1950 was unforgettable. But Lee Lorch had only begun to influence us. He remained at Fisk for five years, and through his personal example of teaching and humanity, he succeeded in redirecting the career goals and affecting the lives of many of his students.

Shortly before that first semester’s classes began, I approached the new chairman for employment as a grader in the math department. I was a junior, and I was undecided between majoring in math or chemistry. Dr. Lorch offered me the job of grading for a freshman level class in Elementary Analysis that he would be teaching, and he suggested I attend the first few meetings of the class to get a feel of what he expected the students to do in their work. After the first class meeting, it was clear to me that I would be attending each meeting of the class for the entire semester.

Dr. Lorch conducted this class as he did all of his other courses. He believed that the students could understand the material, not just learn to do it. He was interested in teaching them the why of mathematics in addition to the how. He also watched the reactions of his class intently, and always maintained good rapport. At the first sign that the class was lost, he would stop, repeat, and give more specific examples. He would motivate the students to participate in class discussions, and would often, through his questioning, draw out of the class the proof of some general result. The students saw that he expected them to learn the material, and they felt compelled to live up to his expectations.