Measurements, Error, Significant Digits, Rounding, etc.

The numbers dealt with in mathematics are generally exact. For example, when a mathematician writes 2 he/she means exactly 2 and all subsequent calculations assume that this 2 means exactly two and not the slightest fraction more or less. In physics many of the numbers dealt with come from measurements of physical quantities and these can never be exact but contain some uncertainty or error. In this context, the word “error” does not carry the usual connotation of “mistake”. “Error” refers to the inevitable uncertainty that accompanies all measurements. As such, the words “error” and “uncertainty” will be used synonymously. What follows is a discussion of how to make a measurement, estimate the associated error, properly record the result and how to handle calculations with associated errors.

How to Make Measurements with Analog Devices and Estimate the Associated Error

An analog device can be defined as any device that is not digital, but from a practical point of view it is also any device that requires reading a scale.

Consider the magnified picture of a portion of an accurately ruled meter stick shown in the accompanying diagram. The measured location of the tip of the arrow is clearly between 20.4 cm and 20.5 cm. A more accurate determination of the location of the arrow can be obtained by estimating where between 20.4 cm and 20.5 cm the tip of the arrow lies. In this case, a reasonable estimate would be 20.46 cm. The 2, the 0, the 4, and the 6 in this measurement are all called significant digits because they give trustworthy information about the position of the arrow. Although there is no question about the 2, the 0 and the 4, the 6 represents a significant estimate as to where between 20.4 cm and 20.5 cm the actual position of the arrow lies. Thus, this measurement is good to four significant digits. The fact that the least significant digit was an estimate is characteristic of all measured quantities.

Any statement about the position of the arrow must also indicate the uncertainty associated with the measurement. In this case, it is conceivable that someone might estimate the last digit as a 5 or perhaps as a 7. It is hard to imagine that anyone would estimate the measurement outside the range from 20.45 cm to 20.47 cm. Therefore a reasonable determination of the position of the arrow would be 20.46 cm ± 0.01 cm. It is understood that the least significant digit, the 6, is an estimate which might be off by no more than ±1. To be clear, the 20.46 cm is the recorded measurement and the ± 0.01 cm is its uncertainty.
Some routinely estimate the uncertainty as half the smallest division of the scale. In this case, since the smallest division is 1 mm (0.1 cm), the uncertainty would be ± 0.05 cm. Although this simple rule eliminates the need for any real thinking on the part of the measurer, it consistently overestimates the actual uncertainty. It assumes that the measurer is incapable of mentally dividing the interval between 20.4 cm and 20.5 cm into 10 equal smaller intervals and deciding to which interval the arrow most closely points and estimate reasonable limits into which the actual measurement must be bound.

Furthermore, measurements recorded as 20.457 cm ± 0.001 cm, or 20.4573 cm ± 0.0001 cm are absurd since it is inconceivable that any measurer is capable of mentally dividing the interval between 20.4 cm and 20.5 cm into 100 or 1000 equal smaller divisions and deciding to which interval the arrow most closely points and estimating reasonable limits of uncertainty. In this case, it is easy to mentally divide the interval between 20.4 cm and 20.5 cm in half and determine that the arrow falls in the interval between 20.45 cm and 20.50 cm. Now all that is required is to mentally divide the interval between 20.45 cm and 20.50 cm into 5 equal smaller intervals and estimate to which interval the arrow most closely points. If the arrow appears to the measurer to be very close to but slightly greater than 20.45 cm, then 20.46 cm is a reasonable estimate. The further from 20.45 cm the arrow appears to the measurer the more likely that 20.47 is a more reasonable estimate. If the arrow appears to the measurer to be closer to 20.50 cm than 20.45 cm then 20.48 cm or 20.49 cm would be reasonable estimates. The point is that this last digit is an estimate and is subject to the discretion of the measurer. The estimated uncertainty is based on how much confidence the measurer has in his estimate and where other measurers might conceivable view the position of the arrow. Following this simple procedure, anyone can mentally divide an interval into 10 equal smaller intervals and estimate the position of the arrow and any attempt to mentally subdivide the interval into 100 or 1000 equal parts in order to obtain more than 4 significant digits is beyond the mental capability of most measurers.

**Significant Digits**

The number of significant digits in a numerical value is determined by first writing the numerical value in scientific notation. Once written in scientific notation, all digits are significant. For example:

- $24.07 = 2.407 \times 10^1$ has 4 significant digits;
- $0.01 = 1 \times 10^{-2}$ has 1 significant digit
- $0.0069 = 6.9 \times 10^{-3}$ has 2 significant digits;
- $100.0 = 1.000 \times 10^2$ has 4 significant digits
- $0.500 = 5.00 \times 10^{-1}$ has 3 significant digits;
- $100. = 1 \times 10^2$ has 3 significant digits
- $2300.0 = 2.3000 \times 10^3$ has 5 significant digits;
- $100. = 1 \times 10^2$ has 3 significant digits

**Rounding**

When the number of digits of a numerical value has to be reduced, the remaining least significant digit has to be rounded off in accordance with the following rules:

1. If the digit immediately following the least significant digit to be retained is 0, 1, 2, 3 or 4, then the last digit retained is left unaltered.
2. If the digit immediately following the least significant digit to be retained is 6, 7, 8, 9, or a 5 followed by a non-zero digit, then the last digit retained is incremented by one.
3. If the digit immediately following the least significant digit to be retained is a 5 followed only by zeros or no other digit, then the last digit retained is left unaltered if it is even and it is incremented by one if it is odd.

Examples: 27.64 rounded to the tenth’s place becomes 27.6
107.317 rounded to the hundredth’s place becomes 107.32
68.50001 rounded to the units place becomes 69
1.635 rounded to the hundredth’s place becomes 1.64
11.850 rounded to the tenth’s place becomes 11.8

**Recording Analog Measurements and Results of Calculations**

In the above example, the position of the arrow is best recorded as 20.46 cm ± 0.01 cm. It is important when recording an analog measurement to ensure that the least significant digit of the recorded measurement and the least significant digit of the uncertainty match in their respective decimal positions. This rule assumes that both the measurement and its uncertainty are in the same units. In the above case, the least significant digit of the measurement is the 6 and the least significant digit of the uncertainty is the 1. Since the 6 and the 1 are both in the same decimal positions (hundredths place), the value is recorded properly. This same rule applies to recording the results of calculations with numbers which have uncertainty. In addition to the above rule, three other general rules also apply and are summarized below:

1. The decimal position of the least significant digit of the recorded measurement and the least significant digit of the uncertainty must match.
2. The uncertainty must NEVER contain more than 2 significant digits.
3. Both the recorded measurement and its uncertainty MUST have units (unless the quantity is dimensionless) and the units MUST be the same.
4. The reported uncertainty is also one standard deviation unless otherwise indicated.

For example, all of the following are improperly recorded except the last.

113.05 cm ± 0.01 mm  769.34 ± .02 nm  57.8 m ± 0.03 m  4.138 x 10^3 cm ± 0.002 cm
45.612 mm ± 0.395 mm  9.12 μm ± 0.057 μm  83.117 ± 0.003 mm  69.41 μF ± 0.08 μF

**NOTE:** Another method used to report uncertainty in the scientific literature is to indicate the uncertainty in parentheses after the measurement. For example, 20.46(1) cm means the 6 is uncertain by ±1. This notation is not the norm and will not be how you are expected to report uncertainty in this lab.

**Recording Digital Measurements**

Digital measurements are easier in the sense that the digits to record for the measurement are visibly provided in the digital display of the measuring device and no estimation of the next digit is possible. The uncertainty in the measurement must be obtained from the specifications of the measurement device. For example, the digital voltmeters used in this lab specify that the uncertainty in the voltage ΔV is ± 0.5% of the voltage reading plus 2 digits in the last decimal place of the reading. The accuracy specifications consist of two parts because the 0.5% reflects the accuracy of the measurement circuit within the voltmeter and the + 2 digits reflects the accuracy of the analog to digital converter within the voltmeter. Thus, for a reading of V = 12.3 volts the associated uncertainty (ΔV) would be (12.3 volts)(0.005) + 0.2 volts = 0.2615 volts. The properly recorded measurement would be 12.3 volts ± 0.3 volts. The rules to follow for recording the results of measurements taken with digital devices are:

1. The uncertainty in a measurement taken with a digital device is calculated from the information provided in the specifications of the device.
2. Record only one digit in the uncertainty.
3. Both the recorded measurement and its uncertainty MUST have units (unless the quantity is dimensionless) and the units MUST be the same.
4. The reported uncertainty is also one standard deviation unless otherwise indicated.

**Calculations Involving Measurements with Specified Uncertainties**

Consider the measured quantities $a$ and $b$ and their associated uncertainties $\Delta a$ and $\Delta b$ given by $a \pm \Delta a$ and $b \pm \Delta b$. The following rules apply when calculating with these numbers.

1. **Adding:** $c = a + b$ and $\Delta c = \Delta a + \Delta b$  
   result: $c \pm \Delta c$
2. **Subtracting:** $c = a - b$ and $\Delta c = \Delta a + \Delta b$ 
   result: $c \pm \Delta c$
3. **Multiplying:** $c = (a)(b)$ and $\frac{\Delta c}{c} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$ 
   result: $c \pm \Delta c$
4. **Dividing:** $c = \frac{a}{b}$ and $\frac{\Delta c}{c} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$ 
   result: $c \pm \Delta c$
5. **Powers:** $c = a^n$ and $\frac{\Delta c}{c} = n \frac{\Delta a}{a}$ 
   result: $c \pm \Delta c$

6. Numbers in formulas which are not measured quantities like $\pi$ have no uncertainty, $\Delta \pi = 0$.
7. Do NOT round in the middle of the calculation. Keep all digits of the calculation until the end of the calculation and then record the result using the general rules given in the section titled Recording Analog Measurements and Results of Calculations.

The above rules are easily generalized when adding, subtracting, multiplying or dividing more than two numbers. The following are examples of rules 1-5 above.

If: $a = 0.76 \text{ cm} \pm 0.08 \text{ cm}$ and $b = 0.51 \text{ cm} \pm 0.07 \text{ cm}$
then $a + b = 1.27 \text{ cm} \pm 0.15 \text{ cm}$ and $a - b = 0.40 \text{ cm} \pm 0.15 \text{ cm}$

If: $A = lw$ and $l = 3.625 \text{ m} \pm 0.001 \text{ m}$, $w = 1.500 \text{ m} \pm 0.001 \text{ m}$, then $A = 5.438 \text{ m} \pm 0.005 \text{ m}$

If: $R = \frac{V}{I}$ and $V = 8.75 \text{ volt} \pm 0.04 \text{ volt}$, $I = 0.125 \text{ A} \pm 0.003 \text{ A}$, the $R = 700. \Omega \pm 20. \Omega$
If $A = \pi r^2$ and $r = 10.75 \text{ cm} \pm 0.05 \text{ cm}$, then $A = 363 \text{ cm}^2 \pm 3 \text{ cm}^2$

**Calculations Involving Measurements with Unspecified Uncertainties**

As an example, consider the measured current $I$ recorded from a digital ammeter as $I = 125 \text{ mA}$. Without having the specifications of the ammeter available, no uncertainty can be calculated and recorded. It would be tempting to assume that the uncertainty is half the least significant digit, that is $\pm 0.5 \text{ mA}$ were upon the rules for adding, subtracting, multiplying, dividing and powers could be applied. However, this assumption may be quite erroneous. Instead certain general rules apply when calculating with measurements with unspecified uncertainties.

1. **Addition and Subtraction:** The decimal position of the least significant digit of the sum or difference must correspond to the decimal position of the least significant digit of the least accurate of the numbers being added or subtracted.
For example: 2.456 + 0.5 + 3.35 = 6.306 recorded as 6.3

The least accurate number is the 0.5 and its least significant digit is in the tenth’s place. Thus the sum is rounded to the tenth’s place. Do NOT round each number to the tenth’s place first and then add, this would give the wrong answer 2.5 + 0.5 + 3.4 = 6.4. Keep all digits and round the final answer appropriately.

2. Multiplication and Division: The product or quotient should have the same number of significant digits as the number with the least number of significant digits being multiplied or divided. In some cases, an additional digit may be retained if dictated by common sense.

For example: (1.369)(4.728) = 6.472632 recorded as 6.473
(105)(37) = 3885 recorded as 3.9 x 10^3
(69.71)/(0.5136) = 135.7281931... recorded as 135.73
(2.54)^3 = 16.387064 recorded as 16.39

3. Keep all digits in the calculation and round the answer appropriately in accordance with the two rules above.

4. Numbers in formulas which are not measured quantities have an infinite number of significant digits.

**Error Specifications**

Consider two measured values a and b of the same quantity. Without knowing the true or exact value, the error is specified using the percent difference defined as follows:

\[
\text{Percent Difference} = \left( \frac{a-b}{(a+b)/2} \right) \times 100 \%
\]

where \(| |\) is the absolute value sign

Consider the measured value a of a quantity whose exact or accepted value is known to be t, then the error is specified using the percent error defined as follows:

\[
\text{Percent Error} = \left( \frac{a-t}{t} \right) \times 100 \%
\]

where again \(| |\) is the absolute value sign

When recording the percent difference or percent error never record more than two significant digits.

**Comparing Measured Values**

Consider comparing two measured values a ± Δa and b ± Δb of the same quantity. Although the percent difference is a quantity which indicates the relative error between the measurements, it does not take into account the uncertainties of each measurement. When the uncertainties Δa and Δb are not known, then percent difference and percent error are the only methods that can be used to compare the values. However, then the uncertainties Δa and Δb are known it is more important to ask whether the two measured values agree to within their measured uncertainties. The procedure for doing this is best illustrated with an example.

Suppose the resistance of a resistor is measured using two different techniques yielding the following two results R_1 = 205.67 Ω ± 0.35 Ω and R_2 = 204.86 Ω ± 0.47 Ω. Do the two measurements agree to within experimental error? Based on its uncertainty, R_1 could range from R_1 - ΔR_1 to R_1 + ΔR_1 or 205.32 Ω to 206.02 Ω. Similarly, R_2 could range from 204.39 Ω to 205.33 Ω. If these two ranges overlap, then the
two measurements agree to within experimental error. In this case they agree since the upper range of $R_2$ overlaps with the lower range of $R_1$. If the ranges do not overlap then there is no experimental agreement.

**Estimated Uncertainties with Repeated Measurements**

The uncertainties discussed so far involve estimating the error in a single measurement. For example, the position of the error on the first page was determined to be 20.46 cm ± 0.01 cm. Repeating this measurement gives no new information since it will undoubtedly yield the same result. However, many measurements involve uncertainties that are more difficult to estimate than those associated with reading a scale. For example, when measuring a time interval using a stop watch, the main source of uncertainty is not the difficulty in reading the dial but is in the variable and unknown reaction time in starting and stopping the watch. Since reaction times will vary, repeating the measurement will undoubtedly yield a different result. These kinds of uncertainties can be estimated by repeating the measurement several times. If it is assumed that the results for any measurement are due to small random errors then the results will be governed by the normal or Gaussian distribution and the following rules apply:

1. For $N$ measurements $x_1$, $x_2$, $x_3$, … $x_N$ of some quantity $x$, the best estimate of the measurement $x$ is given by

   $$ar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$$

2. The uncertainty $\Delta\bar{x}$ of the mean and the standard deviation of the mean $\sigma$ given by

   $$\sigma = \frac{\sigma_x}{\sqrt{N}} \quad \text{where} \quad \sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}}$$

3. The measurement is reported as $\bar{x} \pm \sigma$. The uncertainty in $\bar{x}$, $\Delta\bar{x}$, is the standard deviation of the mean $\sigma$.

For example: Consider the following measurements of the distance from a lens to the focused image by ten different people. The variability arises from the individual determinations of when the image is in focus.

26.41 cm, 23.93 cm, 25.12 cm, 24.64 cm, 22.76 cm, 23.85 cm, 25.10 cm, 23.97 cm, 25.39 cm, 25.48 cm

The uncertainty in each measurement is ±0.03 cm.

The mean $\bar{x} = 24.665$ cm and the standard deviation $\sigma_x = \sqrt{\frac{(9.94825\,\text{cm}^2)}{9}} = 1.05136 \ldots \text{cm}$

Standard deviation of the mean $\sigma = \frac{\sigma_x}{\sqrt{N}} = 0.332 \cdots \text{cm}$ and the result is recorded as 24.7 cm ± 0.3 cm.