
Submissions: This assignment is due in class on Tuesday February 6th, 2018. Each student must submit his or her own assignment. Solutions can either be typed in Latex, MSWord or other such word processing software, or printed clearly. You must write your name and UUID clearly on your submitted assignment.

Academic Integrity: You are encouraged to work in groups, but everyone must write out their own solutions. Absolutely no word to word copying is allowed. Please refer to the course policies and schedules about this. If you have worked with other students on the assignment or referred to external sources, please mention all names and sources on your assignment.

Partial solutions: If you are sure that you know how to arrive at a solution, but you get stuck in some place, it is better to write the partial solution. Honest attempts at partial solutions will be awarded.

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**Problem 1 [30 pts]:** The following are basic results in number theory that you may recall:

(I) *Every odd integer is either of the form $4k + 1$ or $4k + 3$, i.e., it leaves a remainder of 1 or a remainder of 3 when divided by 4, and,*

(II) *Every positive integer can be expressed as the product of prime numbers.* Of course, in the representation there could be a single prime in the product if the number is itself prime, or there could be repeated prime numbers, for example $9 = 3 \cdot 3$, where the $\cdot$ is multiplication.

Using the above, show that there are infinitely many primes of the form $4k + 3$, i.e., there are infinitely many prime numbers that leave a remainder of 3 when divided by 4. What proof strategy did you use?

*A few such primes are 3, 7, 11, 19, 23, 31, ….*

**Problem 2 [20 pts]:** Imagine a party with $n$ people. When a person comes to the party they go and shake hands with a few other people (but not necessarily everyone — in fact there can be unfriendly people who do not shake hands with anyone!). None of the attendees are narcissistic enough to shake hands with themselves. Prove that there are two people who shake the same number of hands.

*For example, suppose 3 people attend the party and everyone shakes hands with everyone. Well, then all 3 shake 2 hands each. If, on the other hand one of them is unfriendly and does not shake hands with anyone but the other two are not, then there are 2 people who shake 1 hand each. Similarly, if all of them are unfriendly, then there are 3 people each shaking 0 hands each. In all the cases, the claim is true. You have to prove this in general—not necessarily by a case analysis.*

**Problem 3 [10 pts]:** Prove that there is no positive integer $n$ such that $n^2 + n^3 = 100$.

**Problem 4 [20 pts]:** Prove that between every two rational numbers there is an irrational number.

**Problem 5 [20 pts]:** Show that if $r$ is an irrational number, there is a unique integer $n$ such that the distance between $r$ and $n$ is less than $1/2$. 