Abstract

Our first aim in this note is to prove some inequalities relating the eigenvalues of a Hermitian matrix with the eigenvalues of its principal matrices induced by a partition of the index set. One of these inequalities extends an inequality proved by Hoffman in [9].

Secondly, we apply our inequalities to estimate the eigenvalues of the adjacency matrix of a graph, and prove, in particular, that for every \( r \geq 3 \), \( c > 0 \) there exists \( \beta = \beta(c, r) \) such that for every \( K_r \)-free graph \( G = G(n, m) \) with \( m > cn^2 \), the smallest eigenvalue \( \mu_n \) of \( G \) satisfies

\[
\mu_n \leq -\beta n.
\]

Similarly for every \( r \geq 3 \), \( c < 1/2 \) there exists \( \gamma = \gamma(c, r) \) such that for every graph \( G = G(n, m) \) with \( m < cn^2 \) and independence number \( \alpha(G) < r \), the second eigenvalue \( \mu_2 \) of \( G \) satisfies

\[
\mu_2 > \gamma n
\]

for sufficiently large \( n \).

1 Introduction

Given an \( m \times n \) matrix \( A \) and nonempty sets \( I \subseteq [m], J \subseteq [n] \), we denote by \( A[I, J] \) the submatrix of the entries \( a_{ij} \) of \( A \) with \( i \in I \) and \( j \in J \); we set \( A_I = A[I, I] \). For every \( n \times n \) matrix \( A \) we denote by \( \mu_1(A), \ldots, \mu_n(A) \) its spectrum; if \( A \) has only real eigenvalues we index them in decreasing order:

\[
\mu_{\max}(A) = \mu_1(A) \geq \ldots \geq \mu_n(A) = \mu_{\min}(A).
\]

Let \( A \) be an \( n \times n \), \( B \) an \( m \times m \) matrix, and \( A \) and \( B \) have only real eigenvalues. As usual we say that the eigenvalues of \( A \) and \( B \) are interlaced if for every \( i = 1, \ldots, m \) the inequalities

\[
\mu_i(A) \geq \mu_i(B) \geq \mu_{n-m+i}(A)
\]
hold. The interlacing is called tight if there exists an integer \( k (0 \leq k \leq m) \) such that

\[ \mu_i (A) = \mu_i (B) \text{ for } 0 \leq i \leq k \text{ and } \mu_{n-m+i} (A) = \mu_i (B) \text{ for } k + 1 \leq i \leq m. \]

Our graph-theoretic notation is standard (see e.g., [2]). For simplicity, all graphs are assumed to be defined on the vertex set \([n] = \{1, \ldots, n\}\). The eigenvalues of a graph \( G \) are the eigenvalues of its adjacency matrix.

Haemers [6] used interlacing techniques to estimate eigenvalues of graphs (see his survey paper [8] for more detailed exposition of the topic and [5] for further development). In this note we use these methods to prove some new inequalities and improve some others. In particular, we show that if \([n] = N_1 \cup \ldots \cup N_k\) is a partition of the index set into nonempty sets and \( A \) is a Hermitian matrix of size \( n \) then for all integers \( m_1, \ldots, m_k \) with \( 0 \leq m_j < |N_j| \),

\[
\mu_1 (A) + \ldots + \mu_{k-1} (A) + \mu_{n-m_1-\ldots-m_k} (A) \geq \sum_{i=1}^k \mu_{|N_i|-m_i} (A_{N_i})
\]

and

\[
\mu_{m_1+\ldots+m_k+1} (A) + \mu_{n-k+1} (A) + \ldots + \mu_n (A) \leq \sum_{i=1}^k \mu_{m_i+1} (A_{N_i}).
\]

The latter was stated for real symmetric matrices by Hoffman in [9]; however, his induction argument is based on a result of Aronszajn [1] that is not readily extendable to a Hermitian \( A \). Moreover, the result used by Hoffman is stronger than the one actually proved by Aronszajn in [1]. We do not attempt to fill this gap - our approach is direct and self-contained.

Furthermore, we shall prove that if \( A = (a_{ij}) \) is Hermitian matrix of size \( n \), and \([n] = N_1 \cup \ldots \cup N_k\) is a partition into nonempty sets, then

\[
\mu_1 (A) + \ldots + \mu_{n-k+1} (A) \geq \sum_{r=1}^k \frac{1}{|N_r|} \sum_{i,j \in N_r} a_{ij},
\]

and

\[
\mu_{n-k+1} (A) + \ldots + \mu_n (A) \leq \sum_{r=1}^k \frac{1}{|N_r|} \sum_{i,j \in N_r} a_{ij} - \frac{1}{n} \sum_{i,j \in [n]} a_{ij}.
\]

2 \hspace{1em} Eigenvalues of Hermitian matrices

In the proof of our first result we shall make use of the following theorem of Cauchy (for a proof, see [10], p. 189).

**Theorem 1.** Let \( A \) be a Hermitian matrix and \( A' \) be its proper principal submatrix. Then the eigenvalues of \( A \) and \( A' \) are interlaced. \( \square \)
As usual, we denote by $A^*$ the adjoint of a matrix $A$. We call a partition $[n] = N_1 \cup \ldots \cup N_k$ proper if none of the sets $N_1, \ldots, N_k$ is empty.

**Theorem 2.** Suppose $2 \leq k \leq n$ and let $A$ be a Hermitian matrix of size $n$. For every proper partition $[n] = N_1 \cup \ldots \cup N_k$, we have

$$
\mu_1(A) + \mu_{n-k+1}(A) + \ldots + \mu_n(A) \leq \sum_{i=1}^{k} \mu_1(A_{N_i}).
$$

(1)

**Proof.** For $k = n$ both sides of (1) are equal to $\text{tr}(A)$, the trace of $A$, so we may suppose $k < n$. To simplify the notation we take

$$
N_j = [n_j + 1, \ldots, n_{j+1}],
$$

where $n_1, \ldots, n_{k+1}$ are integers with $0 = n_1 < \ldots < n_j < \ldots < n_{k+1} = n$. Let $(x_1, \ldots, x_n)$ be an eigenvector with eigenvalue $\mu_1(A)$. For every $j \in [k]$, set $y_j = (x_{n_j+1}, \ldots, x_{n_{j+1}})$ and let $y_j' = (\|y_j\|, 0, \ldots, 0) \in \mathbb{C}^{(I_j)}$. For every $j \in [k]$ there is a unitary matrix $B_j$ such that $y_j = B_jy_j'$, as $\|y_j\| = \|y_j'\|$. The $n \times n$ matrix

$$
U = \begin{pmatrix}
B_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & B_k
\end{pmatrix}
$$

is unitary, i.e. $U^{-1} = U^*$, and thereby the matrix

$$
U^*AU = \begin{pmatrix}
B_1^*A[N_1, N_1]B_1 & \ldots & B_1^*A[N_1, N_k]B_k \\
\vdots & \ddots & \vdots \\
B_k^*A[N_k, N_1]B_1 & \ldots & B_k^*A[N_k, N_k]B_k
\end{pmatrix}
$$

has the same spectrum as $A$. Furthermore, denote the entries of $B_s$ by $b_{pq}^{(s)}$ and let $D = (d_{pq}) \in M_k$ be the Hermitian matrix defined by

$$
d_{pq} = \sum_{i=0}^{k-p+1} \sum_{j=0}^{k-q+1} b_{pq}^{(s)} a_{ij} b_{ij}^{(s)}.
$$

In other words, $D$ consists of the upper left corner entries of the blocks of $U^*AU$. We shall prove that $\mu_1(A)$ is an eigenvalue of $D$ with eigenvector $(\|y_1\|, \ldots, \|y_k\|)$. Indeed, for every $s \in [k]$ we have

$$
\sum_{r=1}^{k} d_{sr} \|y_r\| = \sum_{r=1}^{k} \sum_{i=k_r+1}^{k_{r+1}} \sum_{j=k_r+1}^{k_{r+1}} b_{ij}^{(s)} a_{ij} b_{ij}^{(r)} \|y_r\|
$$

$$
= \sum_{i=k_s+1}^{k_{s+1}} \sum_{r=1}^{k} \left( \sum_{j=k_r+1}^{k_{r+1}} a_{ij} x_j \right) = \sum_{i=k_s+1}^{k_{s+1}} b_{ii}^{(s)} \mu_1 x_i
$$

$$
= \mu_1 \|y_s\|.
$$
From Theorem 1, for every $i \in [k]$, we have $\mu_{n-k+i} (A) \leq \mu_i (D)$ and thus,
\[
\mu_1 (A) + \mu_{n-k+1} (A) + \ldots + \mu_n (A) \leq \mu_1 (D) + \ldots + \mu_k (D) = tr (D).
\]
As no diagonal entry of a Hermitian matrix exceeds its largest eigenvalue, we have, for every $i \in [k]$,
\[
d_{ii} \leq \mu_1 (B_i^* A_{N_i} B_i) = \mu_1 (A_{N_i}),
\]
completing the proof of (1).

Let $\{ e_1, \ldots, e_n \}$ be the standard basis in $\mathbb{C}^n$. For every $M \subset [n]$ we write $P_M$ for the orthogonal projection of $\mathbb{C}^n$ on the space $s (M) = \text{span} \{ e_i | i \in M \}$. Let $Q_M : s (M) \to \mathbb{C}^n$ be defined by
\[
Q_M (u) = v, \ v \in P_M^{-1} (u), \text{ and } (v)_i = 0 \text{ for every } i \notin M.
\]
Observe that for every self-adjoint linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$, the operator $T_M = P_M T Q_M$ maps $s (M)$ into $s (M)$ and is self-adjoint; also, if $A$ is the matrix of $T$ then $A_M$ is the matrix of $T_M$.

Now we shall restate Theorem 2 in operator form.

**Theorem 3.** Suppose $2 \leq k \leq n$ and let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a self-adjoint operator. For every proper partition $[n] = N_1 \cup \ldots \cup N_k$, we have
\[
\mu_1 (T) + \mu_{n-k+1} (T) + \ldots + \mu_n (T) \leq \sum_{i=1}^k \mu_1 (T_{N_i}).
\]
We shall prove that Theorem 2 implies a more general assertion.

**Theorem 4.** Suppose $2 \leq k \leq n$ and let $A$ be a Hermitian matrix of size $n$. For every proper partition $[n] = N_1 \cup \ldots \cup N_k$ and all integers $m_1, \ldots, m_k$ with $0 \leq m_j < |N_j|$, we have
\[
\mu_{m_1+\ldots+m_k+1} (A) + \mu_{n-k+1} (A) + \ldots + \mu_n (A) \leq \sum_{i=1}^k \mu_{m_i+1} (A_{N_i}).
\]

**Proof.** Let $T$ be the self-adjointed linear operator corresponding to $A$. As above we take
\[
N_j = [n_j + 1, \ldots, n_{j+1}],
\]
where $n_1, \ldots, n_{k+1}$ are integers with $0 = n_1 < \ldots < n_j < \ldots < n_{k+1} = n$. For every $j \in [k]$, select a sequence of orthogonal eigenvectors $y_{1j}, \ldots, y_{mj} \in \mathbb{C}^{|N_j|}$ to the eigenvalues $\mu_1 (T_{N_j}), \ldots, \mu_{m_j} (T_{N_j})$; if $m_j = 0$ we select the empty sequence. Set $L_j = \text{span} \{ e_{n_j+1}, \ldots, e_{n_{j+1}} \}$, and let $E$ be the set of all vectors $Q_{L_j} \left( y_{rj} \right)$ ($j \in [k], 1 \leq r \leq m_j$); clearly any two distinct members of $E$ are orthogonal and $|E| = m_1 + \ldots + m_j$. Let $P$ be the orthogonal projection of $\mathbb{C}^n$ on the
To complete the proof observe that the eigenvectors to 

\[ \mu \]

hence, For every 

\[ j \in [k] \], every eigenvector of 

\[ T'_{N_j} \] is orthogonal to every 

\[ y_1^{(j)}, \ldots, y_{m_j}^{(j)} \], hence,

\[ \mu_1 \left( T'_{N_j} \right) \leq \mu_{m_j+1} \left( T_{N_j} \right). \]

To complete the proof observe that the eigenvectors to 

\[ \mu_1 \left( T' \right), \mu_{n-k+1} \left( T' \right), \ldots, \mu_n \left( T' \right) \]

belong to \( E^\perp \); hence, from the Courant-Fischer theorem (e.g. see [10], p. 179), we have 

\[ \mu_1 \left( T' \right) \geq \mu_{n+1} \left( T \right) = \mu_{n+1} \left( A \right), \quad \text{and} \quad \mu_{n-i} \left( T' \right) \leq \mu_{n-i} \left( T \right) = \mu_{n-i} \left( A \right) \quad \text{for} \quad i = 0, \ldots, k - 1. \]

Applying inequality (1) to the matrix \( -A \), we see that

\[ \mu_1 \left( A \right) + \ldots + \mu_{k-1} \left( A \right) + \mu_{n-m_1-\ldots-m_k} \geq \sum_{i=1}^{k} \mu_{|N_i| - m_i} \left( A_{N_i} \right) \]

An immediate consequence from Theorem 2 is the following result.

**Corollary 5.** Let \( A \) be the adjacency matrix of a graph \( G \) of order \( n \). Then

\[ \mu_n \left( A \right) + \mu_1 \left( A \right) \leq n - \alpha - 1, \]

where \( \alpha \) is the independence number of \( G \). \( \Box \)

Note that for \( G = \overline{K_n} + K_{n-\alpha} \) we have equality in (2).

Let \( I_n \) be the \( n \times n \) identity matrix. The result below, whose basic idea goes back to Courant and Hilbert [4], was proved by Haemers ([6], [7]) for real \( S \) and symmetric \( A \) but it is easily seen that it holds for \( S \) complex and \( A \) Hermitian as well.

**Theorem 6.** Let the matrix \( S \) of size \( m \times n \) be such that \( S^* S = I_m \) and let \( A \) be a Hermitian matrix of size \( n \) with eigenvalues \( \mu_1 \geq \ldots \geq \mu_n \). Set \( B = S^* AS \) and let \( \eta_1 \geq \ldots \geq \eta_m \) be the eigenvalues of \( B \) and \( v_1, \ldots, v_m \) the respective eigenvectors.

(i) the eigenvalues of \( A \) and \( B \) are interlaced,

(ii) if \( \eta_i = \mu_i \), or \( \eta_i = \mu_{n-m+i} \), then \( B \) has an eigenvector \( u \) corresponding to \( \eta_i \) such that \( Su \) is an eigenvector of \( A \),

(iii) if for some integer \( l \), \( \eta_i = \mu_i \) for \( i = 1, \ldots, l \) (or \( \eta_i = \mu_{n-m+i} \) for \( i = l, \ldots, m \)) then \( S v_i \) is an eigenvector of \( A \) for \( i = 1, \ldots, l \) (respectively for \( i = l, \ldots, m \)),

(iv) if the interlacing is tight then \( SB = AS \). \( \Box \)

We shall use Theorem 6 to derive two simple inequalities for the eigenvalues of Hermitian matrices.
Theorem 7. Suppose \(2 \leq k \leq n\) and let \(A = (a_{ij})\) be a Hermitian matrix of size \(n\). For every proper partition \([n] = N_1 \cup \ldots \cup N_k\) we have

\[
\mu_1(A) + \ldots + \mu_k(A) \geq \sum_{r=1}^{k} \frac{1}{|N_r|} \sum_{i,j \in N_r} a_{ij},
\]
and

\[
\mu_{k+1}(A) + \ldots + \mu_n(A) \leq \sum_{r=1}^{k} \frac{1}{|N_r|} \sum_{i,j \in N_r} a_{ij} - \frac{1}{n} \sum_{i,j \in [n]} a_{ij}.
\]

Proof. Suppose \(A = (a_{ij})\); set 

\[
e_{rs} = \sum_{i \in N_r, j \in N_s} a_{ij}, \quad \text{and} \quad e = \sum_{i,j \in [n]} a_{ij}.
\]

Note that \(e_{11}, \ldots, e_{kk}\) and \(e\) are real numbers.

For every \(i \in [k]\), set \(n_i = |N_i|\); following Haemers ([6] and [7]), define the \(k \times n\) matrix \(S = (s_{ij})\) by

\[
s_{ij} = \begin{cases} 
\frac{1}{\sqrt{n_i}} & j \in N_i, \\
0 & j \notin N_i,
\end{cases}
\]

It is easy to check that \(S^*S = I_k\), the identity matrix of size \(k\); thus, by Theorem 6, the eigenvalues of the matrix \(B = S^*AS\) and \(A\) are interlaced, i.e., for every \(i \in [k]\), we have

\[
\mu_i(A) \geq \mu_i(B) \geq \mu_{n-k+1}(A),
\]

so

\[
\mu_1(A) + \ldots + \mu_k(A) \geq \mu_1(B) + \ldots + \mu_k(B) = \text{tr}(B).
\]

Easy computations show that \(b_{ij} = e_{ij}/\sqrt{n_i n_j}\) for all \(i, j \in [k]\), so (3) follows.

Furthermore, for every \(i \in [k]\) set \(x_i = \sqrt{n_i}\) and let \(x = (x_1, \ldots, x_k)\). Then 

\[
\|x\|^2 = n_1 + \ldots + n_k = n,
\]

and

\[
\langle Bx, x \rangle = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{e_{ij}}{\sqrt{n_i n_j}} \sqrt{n_i} \sqrt{n_j} = e;
\]

thus, \(\mu_1(B) \geq e/\|x\|^2 = e/n\). Hence, from (6),

\[
\mu_n(A) + \ldots + \mu_{n-k+2}(A) \leq \mu_2(B) + \ldots + \mu_k(B) = \text{tr}(B) - \mu_1(B)
\]

\[
\leq \frac{e_{11}}{n_1} + \ldots + \frac{e_{kk}}{n_k} - \frac{e}{n},
\]

and (4) is proved as well.

Observe that for nonnegative matrices \(A = (a_{ij})\) of size \(n\) and with equal row sums, we have

\[
\mu_1(A) = \frac{1}{n} \sum_{i,j \in [n]} a_{ij};
\]

thus, for such matrices (4) implies (1).
3 Graph eigenvalues

The (combinatorial) Laplacian of a graph $G$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of $G$ and $A(G)$ is the adjacency matrix of $G$. Let

$$0 = \lambda_1 (G) \leq ... \leq \lambda_n (G)$$

be the eigenvalues of $L(G)$.

If $V_1, V_2$ are two disjoint subsets of $V(G)$ we denote by $e(V_1, V_2)$ the number of $V_1 - V_2$ edges.

As an easy consequence of Theorem 7 we obtain the following.

**Theorem 8.** Suppose $2 \leq k \leq n$ and let $G$ be a graph of order $n$. For every proper partition $[n] = N_1 \cup ... \cup N_k$ we have

$$\sum_{i=2}^{k} \lambda_i (G) \leq \sum_{1 \leq i < j \leq n} e(N_i, N_j) \left( \frac{1}{|N_i|} + \frac{1}{|N_j|} \right) \leq \sum_{i=0}^{k-1} \lambda_{n-i} (G). \quad (7)$$

**Proof.** For the matrix $L(G) = (l_{ij})$ we immediately see that

$$e_{rr} = \sum_{i,j \in N_r} l_{ij} = e(N_r, [n] \setminus N_r), \text{ and } e = \sum_{i,j \in [n]} l_{ij} = 0.$$ 

Hence, applying Theorem 7 with $A = L(G)$, from (4) and (3) we obtain (7). \qed

Observe that for $k = 2$, from Theorem 8 we obtain the basic inequalities about the size of a cut of a graph $G$, namely that if $V(G) = N_1 \cup N_2$ is a proper partition then

$$\lambda_2 (G) \leq \frac{e(N_1, N_2)}{|N_1| |N_2|} \leq \lambda_n (G).$$

In fact, Theorem 7 implies that this inequality holds also for weighted graphs as well, as in Mohar ([12], p. 234).

Given a graph $G$ with adjacency matrix $A$, set $\mu_i (G) = \mu_i (A)$. Applying Theorem 7 with $k = 2$ to the adjacency matrix of a graph $G$ we obtain the following corollary.

**Corollary 9.** Suppose $G$ is a graph of order $n \geq 2$ and $V(G) = N_1 \cup N_2$ is a proper partition. Then

$$\mu_n (G) \leq \frac{2e(N_1)}{|N_1|} + \frac{2e(N_2)}{|N_2|} - \frac{2e(G)}{n}.$$ 

Fix a graph $G = G(n, m)$ of order $n \geq 2$ and set $V = V(G)$. The function

$$\Phi(G, t) = \min_{U \subseteq V, |U| = t} \left\{ \frac{e(U)}{t} + \frac{e(V \setminus U)}{n-t} - \frac{m}{n} \right\}.$$
has been investigated in [13]; in particular, it was proved that for every \( r \geq 3, c > 0 \) there exists some \( \beta = \beta(c, r) \) such that for every \( K_r \)-free graph \( G = G(n, m) \) with \( e > cn^2 \),

\[
\Phi(G, \lfloor n/2 \rfloor) \leq -\beta n.
\]

This, together with Corollary 9, implies the following.

**Theorem 10.** For every \( r \geq 3, c > 0 \) there exists \( \beta = \beta(c, r) \) such that for every \( K_r \)-free graph \( G = G(n, m) \) with \( m > cn^2 \),

\[
\mu_n(G) \leq -\beta n.
\]

Similar results hold for \( \mu_2(G) \).

**Lemma 11.** Suppose \( G \) is a graph of order \( n \geq 2 \) and \( V(G) = N_1 \cup N_2 \) is a proper partition. Then

\[
\mu_2(G) \geq \frac{e(N_1)}{|N_1|} + \frac{e(N_2)}{|N_2|} - \frac{e(N_1) - e(N_2)}{|N_1||N_2|} - \frac{e(N_1, N_2)}{\sqrt{|N_1||N_2|}}.
\]

**Proof.** Set \( n_i = |N_i| \) for \( i = 1, 2 \); define a \( 2 \times n \) matrix \( S = (s_{ij}) \) by (5). As in the proof of Theorem 7 we obtain that \( \mu_2(G) \geq \mu_2(B) \) where

\[
B = \begin{pmatrix}
2e(N_1)/n_1 & e(N_1, N_2)/\sqrt{n_1n_2} \\
e(N_1, N_2)/\sqrt{n_1n_2} & 2e(N_2)/n_2
\end{pmatrix}.
\]

Hence we see that

\[
\mu_2(G) \geq \frac{e(N_1)}{n_1} + \frac{e(N_2)}{n_2} - \sqrt{\left(\frac{e(N_1)}{n_1} - \frac{e(N_2)}{n_2}\right)^2 + \frac{e^2(N_1, N_2)}{n_1n_2}}
\]

and the result follows.

We write \( \Gamma(u) \) for the set of vertices adjacent to \( u \), and set \( d(u) = |\Gamma(u)| \). As usual \( \alpha(G) \) denotes the independence number of a graph \( G \). Theorem 10 has the following analog for \( \mu_2 \) of graphs with bounded \( \alpha(G) \).

**Theorem 12.** For every \( r \geq 3, c < 1/2 \) there exists \( \gamma = \gamma(c, r) \) such that for every graph \( G = G(n, m) \) with \( m < cn^2 \) and \( \alpha(G) < r \),

\[
\mu_2(G) > \gamma n
\]

for sufficiently large \( n \).

**Proof.** We were not able to derive this theorem from a general matrix theorem — rather, we give a self-contained proof that uses induction on \( r \).

Denote by \( \overline{G} \) the complement of a graph \( G \). Since \( \alpha(G) < r \) if and only if \( \overline{G} \) is \( K_r \)-free, it is sufficient to prove the following assertion.

For every \( r \geq 3, c > 0 \) there exists \( \gamma = \gamma(c, r) \) such that for every \( K_r \)-free graph \( G = G(n, m) \) with \( m > cn^2 \),

\[
\mu_2(\overline{G}) > \gamma n
\]
for sufficiently large $n$.

Observe that the number $C(G)$ of quadrilaterals (4-cycles) of $G$ satisfies

$$2C(G) \geq \sum_{u,v \in V(G), u \neq v} \left( \frac{|\Gamma (u) \cap \Gamma (v)|}{2} \right) \geq \left( \frac{n}{2} \right) \left( \binom{n}{2} \right)^{-1} \sum_{u,v \in V(G), u \neq v} |\Gamma (u) \cap \Gamma (v)|.$$ 

Since

$$\sum_{u,v \in V(G), u \neq v} |\Gamma (u) \cap \Gamma (v)| = \sum_{u \in V(G)} \left( \frac{d(u)}{2} \right) \geq m \left( \frac{2m}{n} - 1 \right),$$

we see that

$$2C(G) \geq m \left( \frac{2m}{n} - 1 \right) \left( \frac{2m}{n} \left( \frac{2m}{n} - 1 \right) - 1 \right).$$

Hence, if $n > 4/c^2$ we have

$$\left( \frac{2m}{n} - 1 \right) \left( \frac{2m}{n} \left( \frac{2m}{n} - 1 \right) - 1 \right) \geq (2cn-1) (2c(2cn-1)-1)$$

$$> 8c^3n^2 - 2c(2c+1)n > 7c^3n^2$$

and thus,

$$\frac{4C(G)}{m} > 14c^3n^2,$$

so, there is an edge $(u,v)$ that is contained in at least $14c^3n^2$ quadrilaterals.

We shall prove that there exist two disjoint sets $V_1 \subset \Gamma (u)$ and $V_2 \subset \Gamma (v)$ with $e(V_1, V_2) > 2c^3n^2$. Set $U = \Gamma (u), W = \Gamma (v)$; for the number $C'$ of quadrilaterals containing the edge $(u,v)$ we have

$$C' = e(U \setminus W, W \setminus U) + e(U \setminus W, U \cap W) + e(W \setminus U, U \cap W) + 2e(U \cap W) \geq 14c^3n^2.$$ 

Thus, one of the following inequalities holds

$$e(U \setminus W, W \setminus U) \geq 2c^3n^2, $$

$$e(U \setminus W, U \cap W) \geq 2c^3n^2, $$

$$e(W \setminus U, U \cap W) \geq 2c^3n^2, $$

$$e(U \cap W) \geq 4c^3n^2.$$ 

If one of the first three inequalities holds then $V_1$ and $V_2$ clearly exist. Observing that for every graph the size of the maximal cut is at least half the graph size, we see that there is a bipartition $U \cap W = V_1 \cup V_2$ with $e(V_1, V_2) > 2c^3n^2$, and this proves our assertion in the fourth case as well.
We may and shall assume that $|V_1| \leq |V_2|$; hence, obviously, $|V_1| > 2c^3 n$. By averaging we see that there is set $U \subset V_2$ with $|U| = |V_1|$ and

$$e(V_1, U) \geq \frac{|V_1|}{|V_2|} e(V_1, V_2) \geq \frac{|V_1|}{n} 2c^3 n^2 \geq 2c^3 |V_1|^2.$$  

Set $N_1 = V_1$, $N_2 = U$, $k = |N_1| = |N_2|$; clearly, $k = |V_1| > 2c^3 n$.

Consider first the case $r = 3$. Then, as $N_1 \subset \Gamma (u)$, $N_2 \subset \Gamma (v)$, and $G$ has no triangles, we have $e (N_1) = e (N_2) = 0$. Therefore, by applying Lemma 11 to the graph $G_1 = G [N_1 \cup N_2]$, we see that

$$\mu_2 (G_1) \geq k - 1 - \frac{k^2 - e (N_1, N_2)}{k} > 2c^3 k - 1$$

$$> 4c^6 n - 1 > c^6 n$$

for $n > c^{-6}/3$; hence, from Theorem 1, $\mu_2 (\overline{G}) > c^6 n$.

Assume the assertion of the theorem holds for $r' < r$. Suppose $e (N_1) > c^3 k^2/3$; then, as $N_1 \subset \Gamma (u)$ is $K_r$-free, we have

$$\mu_2 (G) \geq \mu_2 (G [N_1]) > \gamma \left( \frac{c^3}{3}, r - 1 \right) k > 2c^3 \gamma \left( \frac{c^3}{3}, r - 1 \right) n,$$

if $k$ is sufficiently large; thus, the assertion is proved if either $e (N_1) > c^3 k^2/3$ or $e (N_2) > c^3 k^2/3$. Assume now $e (N_1) \leq c^3 k^2/3$ and $e (N_2) \leq c^3 k^2/3$. Then, by applying Lemma 11 to the graph $G_1 = \overline{G} [N_1 \cup N_2]$, we see that if $n$ is sufficiently large

$$\mu_2 (G_1) \geq k - 1 - 2c^3 k - \frac{c^3}{3} k - \frac{k^2 - e (N_1, N_2)}{k} \geq -c^3 k + 2c^3 k - 1 \geq c^6 n$$

and, since $\mu_2 (\overline{G}) \geq \mu_2 (G_1)$, our proof is completed.

We shall show that Theorem 10 follows from Theorem 12. Indeed Weyl’s inequality (e. g., see [10], p. 181) states that if $A$ and $B$ are two Hermitian matrices of order $n$ then

$$\mu_2 (A) + \mu_n (B) \leq \mu_2 (A + B).$$

Given a $K_r$-free graph $G$ of order $n$ and size $m > cn^2$, let $A = A (\overline{G})$ and $B = A (G)$, so that

$$\mu_2 (\overline{G}) + \mu_n (G) \leq \mu_2 (K_n) = -1.$$ 

Since $e (\overline{G}) < n^2/2 - e (G) < (1/2 - c) n^2$, applying Theorem 12 to the graph $\overline{G}$ we find that

$$\mu_n (B) \leq -1 - \gamma ((1/2 - c), r) n$$

for sufficiently large $n$, and Theorem 10 follows.

In [3] Chung, Graham and Wilson proved a theorem implying that if $G = G(n, m)$ is a graph with $e \geq cn^2$, $\mu_n (G) = o (n)$, and $\mu_2 (G) = o (n)$ then $G$ contains a $K_r$ if $n$ is sufficiently large. Clearly, Theorem 10 and Theorem 12 strengthen this particular, yet important case of their theorem.
References


