Abstract

We extend in a natural way Szemerédi’s Regularity Lemma to abstract measure spaces.

1 Introduction

In this note we extend Szemerédi’s Regularity Lemma (SRL) to abstract measure spaces. Our main aim is to find general conditions under which the original proof of Szemerédi still works. Another extension of SRL to probability spaces was proved by Tao [3], but his results do not imply our most general result, Theorem 13. To illustrate that our approach has some merit, we outline several applications. Some of these applications seem to be tailored to our approach: in particular, we are not aware of any alternative proofs.

Our notation follows [1].

1.1 Measure triples

A finitely additive measure triple or, briefly, a measure triple \((X, \mathcal{A}, \mu)\) consists of a set \(X\), an algebra \(\mathcal{A} \subset 2^X\), and a complete, nonnegative, finitely additive measure \(\mu\) on \(\mathcal{A}\) with \(\mu(X) = 1\). Thus, \(\mathcal{A}\) contains \(X\) and is closed under finite intersections, unions and differences; the elements of \(\mathcal{A}\) are called measurable subsets of \(X\).

Here are some examples of measure triples.

Example 1 Let \(k, n \geq 1\), write \(2^{[n]^k}\) for the power set of \([n]^k\), and define \(\mu^k\) by \(\mu^k(A) = |A|/n^k\) for every \(A \subset [n]^k\). Then \(([n]^k, 2^{[n]^k}, \mu^k)\) is a measure triple.

Note that there is a one-to-one mapping between undirected \(k\)-graphs on the vertex set \([n]\) and subsets \(G \subset 2^{[n]^k}\) such that if \((v_1, \ldots, v_k) \in G\), then \(\{v_1, \ldots, v_k\}\) is a set of size \(k\) and \(G\) contains every permutation of \((v_1, \ldots, v_k)\). In view of this, we shall consider subsets of \(2^{[n]^k}\) as labelled directed \(k\)-graphs (with loops) on the vertex set \([n]\).
Example 2 Let $k \geq 2$, and let $X_1, \ldots, X_k$ be finite nonempty disjoint sets. Write $2^{X_1 \times \cdots \times X_k}$ for the power set of $X_1 \times \cdots \times X_k$, and define $\mu^k$ by $\mu^k(A) = |A|/(|X_1| \cdots |X_k|)$ for every $A \subset X_1 \times \cdots \times X_k$. Then $(X_1 \times \cdots \times X_k, 2^{X_1 \times \cdots \times X_k}, \mu^k)$ is a measure triple.

In general, we consider subsets of $2^{X_1 \times \cdots \times X_k}$ as labelled $k$-partite $k$-graphs with vertex classes $X_1, \ldots, X_k$.

Example 3 Let $k \geq 1$, and let $\mathbb{B}^k$ be the algebra of the Borel subsets of the unit cube $[0, 1]^k$; write $\mu^k$ for the Lebesgue measure on $\mathbb{B}^k$. Then $([0, 1]^k, \mathbb{B}^k, \mu^k)$ is a measure triple.

1.2 SR-systems

Let us introduce the main objects of our study, SR-systems: measure triples with a suitably chosen semi-ring. Here SR stands for “Szemerédi Regularity” rather than “semi-ring”.

Recall that a set system $S$ is called a semi-ring if it is closed under intersection and for all $A, B \in S$, the difference $A \setminus B$ is a disjoint union of a finite number of members of $S$.

A semi-ring $S$ is called $r$-built if for all $A, B \in S$, the difference $A \setminus B$ is a disjoint union of at most $r$ members of $S$; we say that $S$ is boundedly built if it is $r$-built for some $r$.

An SR-system is a quadruple $(X, \mathcal{A}, \mu, S)$, where $(X, \mathcal{A}, \mu)$ is a measure triple and $S \subset \mathcal{A}$ is a boundedly built semi-ring. Clearly the quadruple $(X, \mathcal{A}, \mu, \mathcal{A})$ is the simplest example of an SR-system based on the measure triple $(X, \mathcal{A}, \mu)$.

For the rest of the section, let us fix an SR-system $(X, \mathcal{A}, \mu, S)$.

Given a set system $X$ and $k \geq 1$, let $X^{(k)}$ be the collection of products of $k$ elements of $X$ any two of which are either disjoint or coincide, i.e.,

$$X^{(k)} = \{A_1 \times \cdots \times A_k : A_i \in X \text{ and } A_i \cap A_j = \emptyset \text{ or } A_i = A_j \text{ for all } i, j \in [k]\}.$$ 

The proof of the following lemma is given in Section 4.

Lemma 4 The set system $S^{(k)}$ is a boundedly built semi-ring.

This assertion is used in the following general construction.

Example 5 For $k \geq 1$, set

$$\mathcal{A}^k = \{A_1 \times \cdots \times A_k : A_i \in \mathcal{A} \text{ for all } i \in [k]\}.$$ 

Write $\mathcal{A}(\mathcal{A}^k)$ for the algebra generated by the set system $\mathcal{A}^k$, and $\mu^k$ for the product measure on $\mathcal{A}(\mathcal{A}^k)$. The quadruple $(X^k, \mathcal{A}(\mathcal{A}^k), \mu^k, S^{(k)})$ is an SR-system.

Let us see three particular examples of this construction.
Example 6 For $k \geq 1$ set $G^k(n) = \left( [n]^k, 2^{[n]^k}, \mu^k, (2^{[n]})^{(k)} \right)$, where $\left( [n]^k, 2^{[n]^k}, \mu^k \right)$ is the measure triple defined in Example 1, and $(2^{[n]})^{(k)}$ is the set of all products of $k$ subsets of $[n]$ any two of which are either disjoint or coincide.

Example 7 For $k \geq 1$ set $B^k = \left( [0,1]^k, B^k, \lambda^k, B^{(k)} \right)$, where $\left( [0,1]^k, B^k, \lambda^k \right)$ is the measure triple defined in Example 3, and $B^{(k)}$ is the set of all products of $k$ Borel subsets of $[0,1]$ any two of which are either disjoint or coincide.

Example 8 For $k \geq 1$ set $B^k = \left( [0,1]^k, B^k, \lambda^k, I^{(k)} \right)$, where $\left( [0,1]^k, B^k, \lambda^k \right)$ is the measure triple defined in Example 3, and $I^{(k)}$ is the set of all products of $k$ intervals $[a,b) \subset [0,1]$ any two of which are either disjoint or coincide.

Example 9 Suppose $k \geq 2$ and $X_1, \ldots, X_k$ are finite nonempty disjoint sets. Set

$$PG(X_1, \ldots, X_k) = \left( X_1 \times \cdots \times X_k, 2^{X_1 \times \cdots \times X_k}, \mu^k, \mathbb{P}(X_1, \ldots, X_k) \right),$$

where

$$\mathbb{P}(X_1, \ldots, X_k) = \{ A_1 \times \cdots \times A_k : A_i \subseteq X_i \text{ for all } i \in [k] \}$$

and $(X_1 \times \cdots \times X_k, 2^{X_1 \times \cdots \times X_k}, \mu^k)$ is the measure triple defined in Example 2. Then $PG(X_1, \ldots, X_k)$ is an SR-system.

1.2.1 Extending $\varepsilon$-regularity

The primary goal of introducing SR-systems is to extend the concept of $\varepsilon$-regular pairs of Szemerédi [4] (see also [1]). For every $A, V \in \mathbb{A}$ set

$$d(A, V) = \frac{\mu(A \cap V)}{\mu(V)}$$

if $\mu(V) > 0$, and $d(A, V) = 0$ if $\mu(V) = 0$.

Definition 10 Let $0 < \varepsilon < 1$, $V \in \mathbb{S}$, and $\mu(V) > 0$. We call a set $A \in \mathbb{A}$ $\varepsilon$-regular in $V$ if

$$|d(A, U) - d(A, V)| < \varepsilon$$

for every $U \in \mathbb{S}$ such that $U \subseteq V$ and $\mu(U) > \varepsilon \mu(V)$.

Let us see what Definition 10 says about directed $k$-graphs.

Take the SR-system $G^k(n)$. Let $G \in G^k(n)$ be a labelled directed $k$-graph with $V(G) = [n]$, and let $(V_1, \ldots, V_k)$ be an ordered $k$-tuple of disjoint nonempty subsets of $[n]$. Write $e(V_1, \ldots, V_k)$ for the number of edges $(v_1, \ldots, v_k) \in E(G)$ such that $v_i \in V_i$ for $i = 1, \ldots, k$. 

3
Now if $G$ is $\varepsilon^{1/k}$-regular in $V_1 \times \cdots \times V_k$, then, for every ordered $k$-tuple $(U_1, \ldots, U_k)$ such that $U_i \subset V_i$ and $|U_i| > \varepsilon |V_i|$ for $i = 1, \ldots, k$, we obtain

\[
\left| \frac{e(V_1, \ldots, V_k)}{|V_1| \cdots |V_k|} - \frac{e(U_1, \ldots, U_k)}{|U_1| \cdots |U_k|} \right| < \varepsilon.
\]

Note that for $k = 2$ this condition is essentially equivalent to the definition of an "$\varepsilon$-regular pair".

Finally let us define $\varepsilon$-regularity with respect to partitions.

**Definition 11** Let $0 < \varepsilon < 1$ and $\mathcal{P}$ be a partition of $X$ into sets belonging to $\mathbb{S}$. We call a set $A \in \mathbb{A}$ $\varepsilon$-regular in $\mathcal{P}$ if

\[
\sum \{\mu(P) : P \in \mathcal{P}, A \text{ is not } \varepsilon\text{-regular in } P\} < \varepsilon.
\]

### 1.3 Partitions in measure triples

Given a collection $\mathcal{X}$ of subsets of $X$, we write $\Pi(\mathcal{X})$ for the family of finite partitions of $X$ into sets belonging to $\mathcal{X}$. We shall be mainly interested in $\Pi(\mathcal{S})$.

Let $\mathcal{P}, \mathcal{Q}$ be partitions of $X$, and $A \subset X$. We say that $\mathcal{P}$ refines $A$ (in notation $\mathcal{P} \succ A$) if $A$ is a union of members of $\mathcal{P}$, and that $\mathcal{P}$ refines $\mathcal{Q}$ (in notation $\mathcal{P} \succ \mathcal{Q}$) if $\mathcal{P}$ refines each $Q \in \mathcal{Q}$. We write $\mathcal{P} \cap \mathcal{Q}$ for the partition consisting of all nonempty intersections $P \cap Q$, where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$.

Define the partition $\mathcal{P}^k$ of $X^k$ as

\[
\mathcal{P}^k = \left\{ P_{i_1} \times \cdots \times P_{i_k} : P_{i_j} \in \mathcal{P}, \text{ for all } j \in [k] \right\}.
\]

#### 1.3.1 Bounding families of partitions

We say that a family of partitions $\Phi \subset \Pi(\mathcal{S})$ bounds $\Pi(\mathcal{S})$ if for every $\mathcal{P} \in \Pi(\mathcal{S})$, there exists $\mathcal{Q} \in \Phi$ such that $\mathcal{Q} \succ \mathcal{P}$ and $|\mathcal{Q}| \leq \varphi(|\mathcal{P}|)$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a fixed increasing function, the rate of $\Phi$.

Here is an example of a bounding family. Given an integer $k \geq 2$, take the SR-system $(X^k, a(A^k), \mu^k, \mathcal{S}^{(k)})$ and define a family of partitions $\Phi^k \subset \Pi(\mathcal{S}^{(k)})$ as

\[
\Phi^k = \{ \mathcal{F}^k : \mathcal{F} \in \Pi(\mathcal{S}^{(k)}) \}.
\]

**Lemma 12** The family $\Phi^k$ bounds $\Pi(\mathcal{S}^{(k)})$.

### 2 The main result

We are ready now to state our main theorem whose proof is presented in 4.1.
Theorem 13 Let \((X, \mathcal{A}, \mu, S)\) be an SR-system and suppose \(\Phi\) is a family of partitions bounding \(\Pi(S)\). Given a finite collection of measurable sets \(\mathcal{L} \subseteq \mathcal{A}\), a partition \(\mathcal{P} \in \Pi(S)\), and \(\epsilon > 0\), there exists \(q = q(\epsilon, |\mathcal{L}|, |\mathcal{P}|)\) and \(Q \in \Phi\) such that:
- \(Q \supseteq \mathcal{P}\);
- every \(A \in \mathcal{L}\) is \(\epsilon\)-regular in \(Q\);
- \(|Q| \leq q\).

Our next goal is to show that Theorem 13 implies various types of regularity lemmas. We emphasize the three steps that are necessary for its application:
(i) select a measure triple \((X, \mathcal{A}, \mu)\);
(ii) introduce \(\epsilon\)-regularity by fixing a boundedly built semi-ring \(S \subseteq \mathcal{A}\);
(iii) select a bounding family of partitions \(\Phi \subseteq \Pi(S)\) by demonstrating an upper bound on its rate \(\varphi(p)\).

We turn now to specific applications.

3 Examples

To obtain more familiar versions of the Regularity Lemma, we extend the concept of “\(\epsilon\)-equitable partitions” and investigate when such partitions form bounding families.

3.1 Equitable partitions

Given \(\epsilon > 0\) and a measure triple \((X, \mathcal{A}, \mu)\), a partition \(\mathcal{P} = \{P_0, \ldots, P_p\} \in \Pi(\mathcal{A})\) is called \(\epsilon\)-equitable, if \(\mu(P_0) \leq \epsilon\) and \(\mu(P_1) = \cdots = \mu(P_p) \leq \epsilon\).

Let \(k \geq 2\), take the SR-system \((X^k, \mathcal{A}(\mathcal{A}^k), \mu^k, S^{(k)})\), and define a family of partitions \(\Phi^k(\epsilon) \subseteq \Pi(S^{(k)})\) as
\[
\Phi^k(\epsilon) = \{\mathcal{P}^k : \mathcal{P} \in \Pi(\mathcal{A}) \text{ and } \mathcal{P} \text{ is } \epsilon\text{-equitable}\}.
\]

It is possible to prove that under some mild conditions on \((X, \mathcal{A}, \mu)\) the family \(\Phi^k(\epsilon)\) bounds \(\Pi(S^{(k)})\). To avoid technicalities, we prove this claim for the SR-system \(\mathcal{G}^k(n) = \left([n]^k, 2^{[n]k}, \mu^k, (2^{[n]})^{(k)}\right)\). Let \(\Phi^k(n, \epsilon)\) be defined by (1) for the SR-system \(\mathcal{G}^k(n)\).

Lemma 14 Let \(0 < \epsilon < 1\) and \(n > 1/\epsilon\). The family \(\Phi^k(n, \epsilon)\) bounds \(\Pi\left((2^{[n]})^{(k)}\right)\) with rate
\[
\varphi(p) = \left([2/\epsilon] + 1\right)^k 2^{pk^2}.
\]

Likewise, let \(\Phi^k([0, 1], \epsilon)\) be defined by 1 for the SR-system \(\mathcal{B}^k = \left([0, 1]^k, \mathbb{B}^k, \lambda^k, \mathbb{B}^{(k)}\right)\).

Lemma 15 Let \(0 < \epsilon < 1\). The family \(\Phi^k(\epsilon)\) bounds \(\Pi\left(\mathbb{B}^{(k)}\right)\) with rate
\[
\varphi(p) = \left([1/\epsilon] + 1\right)^k 2^{pk^2}.
\]

\[\square\]
3.1.1 Regularity lemmae for \( k \)-graphs

We first state a Regularity Lemma for directed \( k \)-graphs. As noted above we represent directed \( k \)-graphs as subsets of \( 2^{[n]} \) and define regularity in terms of the SR-system

\[
G^k(n) = \left( [n]^k, 2^{[n]^k}, \mu^k, (2^{[n]})^{(k)} \right)
\]

**Theorem 16** For all \( 0 < \varepsilon < 1 \) and positive integers \( k, l \), there exist \( n_0(k, \varepsilon) \) and \( q(k, l, \varepsilon) \) such that if \( n > n_0(k, \varepsilon) \) and \( \mathcal{L} \) is a collection of \( l \) subsets directed \( k \)-graphs on the vertex set \([n]\), then there exists a partition \( Q = \{Q_0, \ldots, Q_q\} \) of \([n]\) satisfying

(i) \( q \leq q(k, l, \varepsilon) \);

(ii) \( |Q_0| < \varepsilon n, |Q_1| = \cdots = |Q_q| < \varepsilon n \);

(iii) Every graph \( G \in \mathcal{L} \) is \( \varepsilon \)-regular in at least \( (1 - \varepsilon)q^k \) sets \( Q_{i_1} \times \cdots \times Q_{i_k} \), where \( (i_1, \ldots, i_k) \) is a \( k \)-tuple of distinct elements of \([q]\).

As a consequence we obtain a Regularity Lemma for undirected \( k \)-graphs. For \( k = 2 \) this is the result of Szemerédi, for \( k > 2 \) this is the result of Chung [2]. Recall that undirected \( k \)-graphs are subsets \( G \subset 2^{[n]} \) such that if \( (v_1, \ldots, v_k) \in G \), then \( \{v_1, \ldots, v_k\} \) is a set of size \( k \) and \( G \) contains each permutation of \( \{v_1, \ldots, v_k\} \).

**Theorem 17** For all \( 0 < \varepsilon < 1 \) and positive integers \( k, l \), there exist \( n_0(k, \varepsilon) \) and \( q(k, l, \varepsilon) \) such that if \( n > n_0(k, \varepsilon) \) and \( \mathcal{L} \) is a collection of \( l \) undirected \( k \)-graphs on the vertex set \([n]\), then there exists a partition \( Q = \{Q_0, \ldots, Q_q\} \) of \([n]\) satisfying:

i) \( q \leq q(k, l, \varepsilon) \);

ii) \( |Q_0| < \varepsilon n, |Q_1| = \cdots = |Q_q| < \varepsilon n \);

iii) For every graph \( G \in \mathcal{L} \), there exist at least \( (1 - \varepsilon)q^k \) sets \( \{i_1, \ldots, i_k\} \) of distinct elements of \([q]\) such that \( G \) is \( \varepsilon \)-regular in \( Q_{i_1} \times \cdots \times Q_{i_k} \) for every permutation of \( \{i_1, \ldots, i_k\} \) of \( \{i_1, \ldots, i_k\} \).

3.1.2 A regularity lemma for \( k \)-partite \( k \)-graphs

Considering the SR-system \( \mathcal{PG}(X_1, \ldots, X_k) \) from Example 9 we obtain a regularity lemma for \( k \)-partite \( k \)-graphs.

**Theorem 18** Let \( X_1, \ldots, X_k \) be disjoint sets with \( |X_1| = \cdots = |X_k| = n \). For all \( 0 < \varepsilon < 1 \) and positive integers \( k, l \), there exist \( n_0(k, \varepsilon) \) and \( q(k, l, \varepsilon) \) such that if \( n > n_0(k, \varepsilon) \) and \( \mathcal{L} \) is a collection of \( l \) undirected \( k \)-partite \( k \)-graphs with vertex classes \( X_1, \ldots, X_k \), then for each \( i \in [k] \), there exist a partition \( Q_i = \{Q_{i,0}, \ldots, Q_{i,q}\} \) of \( X_i \), satisfying:

i) \( q \leq q(k, l, \varepsilon) \);

ii) \( |Q_{i,0}| < \varepsilon n, |Q_{i,1}| = \cdots = |Q_{i,q}| < \varepsilon n \);

iii) For every graph \( G \in \mathcal{L} \), there exist at least \( (1 - \varepsilon)q^k \) vectors \( (i_1, \ldots, i_k) \in [q]^k \) such that \( G \) is \( \varepsilon \)-regular in \( Q_{i_1,i_1} \times \cdots \times Q_{i_k,i_k} \).
3.1.3 A regularity lemma for measurable subsets of the unit cube

Now define regularity according to the SR-system $\mathcal{B}^k = \left( [0,1]^k, \mathbb{B}^k, \lambda^k, \mathcal{B}^{(k)} \right)$. We obtain the following result.

**Theorem 19** For all $0 < \varepsilon < 1$ and positive integers $k, l$, there exists $q(k, l, \varepsilon)$ such that if $\mathcal{L}$ is a collection of $l$ measurable subsets of the cube $[0,1]^k$, then there exists a partition $Q = \{Q_0, \ldots, Q_q\}$ of $[0,1]$ into measurable sets satisfying:

i) $q \leq q(k, l, \varepsilon)$;

ii) $\mu(Q_0) < \varepsilon$, $\mu(Q_1) = \cdots = \mu(Q_q) < \varepsilon$;

iii) Every set $L \in \mathcal{L}$ is $\varepsilon$-regular in at least $(1 - \varepsilon)q^k$ sets $Q_{i_1} \times \cdots \times Q_{i_k}$, where $(i_1, \ldots, i_k)$ is a $k$-tuple of distinct elements of $[q]$.

Finally let us define regularity according to the SR-system $\mathcal{B}^k = \left( [0,1]^k, \mathbb{B}^k, \lambda^k, \mathcal{B}^{(k)} \right)$.

We obtain a result which we believe is specific to our approach.

**Theorem 20** For all $0 < \varepsilon < 1$ and positive integers $k, l$, there exists $q(k, l, \varepsilon)$ such that if $\mathcal{L}$ is a collection of $l$ measurable subsets of the cube $[0,1]^k$ then there exists a partition $Q = \{Q_0, \ldots, Q_q\}$ of $[0,1]$ satisfying:

i) $q \leq q(k, l, \varepsilon)$;

ii) $\mu(Q_0) < \varepsilon$, and the sets $Q_1, \ldots, Q_q$ are intervals of equal length $< \varepsilon$;

iii) Every set $L \in \mathcal{L}$ is $\varepsilon$-regular in at least $(1 - \varepsilon)q^k$ bricks $Q_{i_1} \times \cdots \times Q_{i_k}$, where $(i_1, \ldots, i_k)$ is a $k$-tuple of distinct elements of $[q]$.

4 Proofs

4.1 Proof of Theorem 13

Our proof is an adaptation of the original proof of Szemerédi [4] (see also [1]). The following basic lemma is known as the “defect form of the Cauchy-Schwarz inequality”; for a proof see [1].

**Lemma 21** Let $x_i$ and $c_i$ be positive numbers for $i = 1, \ldots, n$. Then

$$\sum_{i=1}^{n} c_i \sum_{i=1}^{n} c_i x_i^2 \geq \left( \sum_{i=1}^{n} c_i x_i \right)^2.$$

If $J$ is a proper subset of $[n]$ and $\gamma > 0$ is such that

$$\sum_{i=1}^{n} c_i \sum_{i \in J} c_i x_i \geq \sum_{i=1}^{n} c_i x_i \sum_{i \in J} c_i + \gamma,$$

then

$$\sum_{i=1}^{n} c_i \sum_{i=1}^{n} c_i x_i^2 \geq \left( \sum_{i=1}^{n} c_i x_i \right)^2 + \gamma^2 \left( \sum_{i \in J} c_i \sum_{i \in [n] \setminus J} c_i \right). \quad \square$$
Let \( \mathcal{P} = \{P_1, \ldots, P_p\} \in \Pi(S) \), and \( A \in \mathbb{A} \). Define the index of \( \mathcal{P} \) with respect to \( A \) as

\[
\text{ind}_A \mathcal{P} = \sum_{P_i \in \mathcal{P}} \mu(P_i) d^2(A \cap P_i).
\]

Note that for every \( A \in \mathbb{A} \),

\[
\text{ind}_A \mathcal{P} = \sum_{P_i \in \mathcal{P}} \mu(P_i) d^2(A \cap P_i) \leq \sum_{P_i \in \mathcal{P}, \mu(P_i) > 0} \frac{\mu(A \cap P_i) \mu(P_i)}{\mu(P_i)} = \mu(A) \leq 1. \quad (2)
\]

**Lemma 22** If \( \mathcal{P}, \mathcal{Q} \in \Pi(S) \), \( A \in \mathbb{A} \), and \( \mathcal{Q} \supset \mathcal{P} \) then \( \text{ind}_A \mathcal{Q} \geq \text{ind}_A \mathcal{P} \).

**Proof** For simplicity we shall assume that \( \mathcal{P} \) and \( \mathcal{Q} \) consist only of sets of positive measure. Fix \( P \in \mathcal{P} \) and for every \( Q_i \subset P \), set

\[
c_i = \mu(Q_i) \quad \text{and} \quad x_i = d(A \cap Q_i).
\]

Note that

\[
\sum_{Q_i \subset P} c_i = \sum_{Q_i \subset P} \mu(Q_i) = \mu(P) \quad \text{and} \quad \sum_{Q_i \subset P} c_i x_i = \mu(A \cap P).
\]

The Cauchy-Schwarz inequality (the first part of Lemma 21) implies that

\[
\sum_{Q_i \subset P} \mu(Q_i) d^2(A \cap Q_i) = \sum_{Q_i \subset P} c_i x_i^2 \geq \frac{1}{\mu(P)} \left( \sum_{Q_i \subset P} c_i x_i \right)^2 = \frac{\mu^2(A \cap P)}{\mu(P)}.
\]

Summing over all sets \( P \in \mathcal{P} \), the desired inequality follows. \( \square \)

The next lemma supports the proof of Lemma 24.

**Lemma 23** Suppose \( A, S, T \in \mathbb{A} \), \( T \subset S \) and \( \mu(T) > 0 \). If

\[
|d(A \cap T) - d(A \cap S)| \geq \epsilon \quad \text{(3)}
\]

then every partition \( \mathcal{U} = \{U_1, \ldots, U_p\} \in \Pi(A) \) such that \( \mathcal{U} \supset S \) and \( \mathcal{U} \supset T \), satisfies

\[
\sum_{U_i \subset S} \mu(U_i) d^2(A, U_i) \geq \mu(S) d^2(A, S) + \epsilon^2 \mu(T).
\]

**Proof** Let the partition \( \mathcal{U} = \{U_1, \ldots, U_p\} \in \Pi(A) \) be such that \( \mathcal{U} \supset S \) and \( \mathcal{U} \supset T \). For every \( U_i \subset S \), set

\[
c_i = \mu(U_i), \quad x_i = d(A, U_i),
\]

and observe that

\[
\sum_{U_i \subset S} c_i = \sum_{U_i \subset S} \mu(U_i) = \mu(S) \quad \text{and} \quad \sum_{U_i \subset S} c_i x_i = \sum_{U_i \subset S} \mu(A \cap U_i) = \mu(A \cap S).
\]
Similarly, we have
\[ \sum_{U_i \subseteq T} c_i = \mu(T) \quad \text{and} \quad \sum_{U_i \subseteq T} c_i x_i = \sum_{U_i \subseteq T} \mu(A \cap U_i) = \mu(A \cap T). \]

Inequality (3) implies that either
\[ d(A, T) > d(A, S) + \epsilon \quad \text{(4)} \]

or
\[ d(A, S) > d(A, T) + \epsilon. \]

Assume that (4) holds; the argument in the other case is identical. Hence, \( \mu(T) \neq \mu(S) \), so \( T \subset S \) implies that \( \mu(T) < \mu(S) \). Furthermore,
\[
\sum_{U_i \subseteq T} c_i x_i = d(A, T) \mu(T) > (d(A, S) + \epsilon) \mu(T) = (d(A, S) + \epsilon) \sum_{U_i \subseteq T} c_i \]
\[
= \sum_{U_i \subseteq S} c_i x_i + \epsilon \sum_{U_i \subseteq T} c_i.
\]

By the definition of \( c_i \) and \( x_i \), we have
\[
\sum_{U_i \subseteq S} \mu(U_i) d^2(A, U_i) = \sum_{U_i \subseteq S} c_i x_i^2.
\]

Therefore, setting
\[
\lambda = \epsilon \sum_{U_i \subseteq S} c_i \sum_{U_i \subseteq T} c_i = \epsilon \mu(T) \mu(S),
\]

and applying the second part of Lemma 21, we find that
\[
\mu(S) \sum_{U_i \subseteq S} \mu(U_i) d^2(A, U_i) \geq \left( \sum_{U_i \subseteq S} c_i x_i \right)^2 + \lambda^2 / \left( \sum_{U_i \subseteq T} c_i \sum_{U_i \subseteq S \setminus T} c_i \right)
\]
\[
= \mu^2(A \cap S) + \epsilon^2 \frac{\mu^2(S) \mu(T)}{\mu(S) - \mu(T)}.
\]

Hence,
\[
\sum_{U_i \subseteq S} \sum_{U_i \subseteq S} \mu(U_i) d^2(A, U_i) \geq \mu(S) d^2(A, S) + \epsilon^2 \frac{\mu(T)}{\mu(S) - \mu(T)} > \mu(S) d^2(A, S) + \epsilon^2 \mu(T)
\]

and this is exactly the desired inequality. \( \square \)

The following lemma gives a condition for an absolute increase of \( \text{ind}_A \mathcal{P} \) resulting from refining.

**Lemma 24** Let \( 0 < \epsilon < 1 \) and \( S \) be \( r \)-built. If \( \mathcal{P} \in \Pi(S) \) and \( A \in \mathbb{A} \) is not \( \epsilon \)-regular in \( \mathcal{P} \) then there exists \( \mathcal{Q} \in \Pi(S) \) satisfying \( \mathcal{Q} \triangleright \mathcal{P} \), \( |\mathcal{Q}| \leq (r + 1) |\mathcal{P}| \), and
\[
\text{ind}_A \mathcal{Q} \geq \text{ind}_A \mathcal{P} + \epsilon^4. \quad \text{(5)}
\]
Proof Let $\mathcal{P} = \{P_1, \ldots, P_p\}$ and $\mathcal{N}$ be the set of all $P_i$ for which $A$ is not $\epsilon$-regular in $P_i$. Since $A$ is not $\epsilon$-regular in $\mathcal{P}$, by definition, we have

$$\sum_{P_i \in \mathcal{N}} \mu(P_i) \geq \epsilon.$$ 

For every $P_i \in \mathcal{N}$, since $A$ is not $\epsilon$-regular in $P_i$, there is a set $T_i \subset P_i$ such that $T_i \in \mathcal{S}$, $\mu(T_i) > \epsilon \mu(P_i)$, and

$$|d(A, P_i) - d(A, T_i)| \geq \epsilon.$$

Since $\mathcal{S}$ is $r$-built, for every $P_i \in \mathcal{N}$, there is a partition of $P_i \setminus T_i$ into $r$ disjoint sets $A_{i_1}, \ldots, A_{i_s} \in \mathcal{S}$; hence $\{A_{i_1}, \ldots, A_{i_s}, T_i\}$ is a partition of $P_i$ into at most $(r + 1)$ sets belonging to $\mathcal{S}$. Let $\mathcal{Q}$ be the collection of all sets $A_{i_1}, \ldots, A_{i_s}, T_i$, where $P_i \in \mathcal{N}$, together with all sets $P_j \in \mathcal{P} \setminus \mathcal{N}$. Clearly $\mathcal{Q} \in \Pi(\mathcal{S})$; also, $\mathcal{Q} \supset T_i$ and $\mathcal{Q} \supset P_i$ for every $P_i \in \mathcal{N}$, and

$$|\mathcal{Q}| \leq (r + 1)|\mathcal{N}| + |\mathcal{P}| - |\mathcal{N}| \leq (r + 1)|\mathcal{P}|.$$

Thus, to finish the proof, we have to prove (5). Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$. For every $P_k \in \mathcal{N}$, Lemma 23 implies that

$$\sum_{Q_i \subset P_k} \mu(Q_i)d^2(A, Q_i) \geq \mu(P_k)d^2(A, P_k) + \epsilon^2 \mu(T_k) \geq \mu(P_k)d^2(A, P_k) + \epsilon^3 \mu(P_k). \tag{6}$$

For any $P_k \in \mathcal{P}$, Lemma 22 implies that

$$\sum_{Q_i \subset P_k} \mu(Q_i)d^2(A, Q_i) \geq \mu(P_k)d^2(A, P_k). \tag{7}$$

Now, by (6) and (7), we obtain

$$\text{ind}_A \mathcal{Q} = \sum_{Q_i \subset \mathcal{Q}} \mu(Q_i)d^2(A, Q_i) \geq \sum_{P_k \in \mathcal{P}} \mu(P_k)d^2(A, P_k) + \epsilon^3 \sum_{P_i \in \mathcal{N}} \mu(P_k)$$

$$\geq \sum_{P_k \in \mathcal{P}} \mu(P_k)d^2(A, P_k) + \epsilon^4 = \text{ind}_A \mathcal{P} + \epsilon^4,$$

completing the proof. \qed

Proof of Theorem 13 Suppose $\mathcal{S}$ is $r$-built, $\Phi$ bounds $\Pi(\mathcal{S})$ with rate $\varphi(\cdot)$ and let $|\mathcal{P}| = p$. Define a function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\psi(1, p) = p; \tag{8}$$
$$\psi(s + 1, p) = (r + 1)\varphi(\psi(s, p)), \text{ for every } s > 1.$$

We shall show that the partition $\mathcal{Q} \in \Phi$ may be selected so that $|\mathcal{Q}| \leq \psi(l |\epsilon^{-1}|, p)$.

Select first a partition $\mathcal{P}_0 \in \Phi$ such that $\mathcal{P}_0 \supset \mathcal{P}$ and $|\mathcal{P}_0| \leq \varphi(|\mathcal{P}|)$. We build recursively a sequence of partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots$ satisfying

$$\mathcal{P}_{i+1} \supset \mathcal{P}_i, \tag{9}$$
$$|\mathcal{P}_{i+1}| \leq \varphi((r + 1)|\mathcal{P}_i|), \tag{10}$$
$$\exists A_i \in \mathcal{G} : \text{ind}_{A_i} \mathcal{P}_{i+1} \geq \text{ind}_{A_i} \mathcal{P}_i + \epsilon^4 \tag{11}$$

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for every $i = 0, 1, \ldots$ The sequence is built according the following rule: If all $A \in \mathcal{L}$ are $\varepsilon$-regular in $\mathcal{P}_i$, then we stop. Otherwise there exists $A_i \in \mathcal{L}$ that is not $\varepsilon$-regular in $\mathcal{P}_i$. Then, by Lemma 24, there is a partition $\mathcal{P}'_i \in \Pi(S)$ such that

$$
\mathcal{P}'_i \succeq \mathcal{P}_i,
$$

$$
|\mathcal{P}'_i| \leq (r + 1)|\mathcal{P}_i|,
$$

$$
\text{ind}_{A_i}\mathcal{P}'_i \geq \text{ind}_{A_i}\mathcal{P}_i + \varepsilon^4.
$$

Since $\Phi$ bounds $S$ with rate $\varphi$, there is a partition $\mathcal{P}_{i+1} \in \Phi$ such that $\mathcal{P}_{i+1} \succeq \mathcal{P}'_i$ and $|\mathcal{P}_{i+1}| \leq \varphi(|\mathcal{P}'_i|)$. Hence, (9), (10), and (11) hold.

Set $k = \lceil \varepsilon^{-4} \rceil$. If the sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots$ has more than $lk$ terms then, by the pigeon-hole principle, there exist a set $A \in \mathcal{L}$ and a sequence $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_{lk+1}}$, such that

$$
\text{ind}_{A_i}\mathcal{P}_{i_{j+1}} \geq \text{ind}_{A_i}\mathcal{P}_{i_j} + \varepsilon^4
$$

for every $j = 1, \ldots, k$. Hence, we find that

$$
\text{ind}_{A_i}\mathcal{P}_{i_{k+1}} \geq \text{ind}_{A_i}\mathcal{P}_{i_1} + k\varepsilon^4 > k\varepsilon^4 \geq 1,
$$

contradicting (2). Therefore, all $A \in \mathcal{L}$ are $\varepsilon$-regular in some partition $\mathcal{Q} = \mathcal{P}_i$. By (10), $|\mathcal{Q}| \leq \psi(l \lceil \varepsilon^{-4} \rceil, p)$, completing the proof. \qed

### 4.2 Proof of lemma 14

**Proof** Select a partition $\mathcal{P} = \{P_1, \ldots, P_p\} \in \Pi \left((2^{[n]} \choose k)\right)$; for every $i \in [p]$ let

$$
P_i = R_{i1} \times \cdots \times R_{ik}, \ R_{ij} \subset [n], \ \text{for } j \in [k].
$$

Let

$$
\mathcal{R} = \cap_{i \in [p], j \in [k]} \{R_{ij}, X \setminus R_{ij}\}.
$$

and set $r = |\mathcal{R}|$. Clearly, $r \leq 2^{pk}$. Our first goal is to find an $\varepsilon$-equitable partition $\mathcal{Q} \succ \mathcal{R}$ with

$$
|\mathcal{Q}| \leq \left(\frac{2}{\varepsilon} + 1\right)2^{pk}.
$$

Suppose first that $n \geq 2r/\varepsilon$. To construct the required $\mathcal{Q}$, partition every $R \in \mathcal{R}$ into sets of size $\lceil \varepsilon n/r \rceil$ and a smaller residual set. The measure of each member of $\mathcal{Q}$ that is not residual is at most $\lceil \varepsilon n/r \rceil / n \leq \varepsilon$. The total measure of all residual sets is less than

$$
\frac{\lceil \varepsilon n/r \rceil}{n} r \leq \varepsilon,
$$

thus, $\mathcal{Q}$ is an $\varepsilon$-equitable partition refining $\mathcal{R}$. Since

$$
|\mathcal{Q}| \leq \frac{n}{\lceil \varepsilon n/r \rceil} + r \leq \frac{2n}{\varepsilon n/r} + r = \left(\frac{2}{\varepsilon} + 1\right) r \leq \left(\frac{2}{\varepsilon} + 1\right)2^{pk},
$$

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\( \mathcal{Q} \) has the required properties.

Let now \( n < 2r/\varepsilon \) and \( \mathcal{Q} \) be the partition of [\( n \)] into \( n \) sets of size 1. Since \( \varepsilon > 1/n \), the partition \( \mathcal{Q} \) is \( \varepsilon \)-equitable and refines \( \mathcal{R} \). Since

\[
|\mathcal{Q}| = n < \frac{2}{\varepsilon} r \leq \left( \frac{2}{\varepsilon} + 1 \right) 2^{pk},
\]

\( \mathcal{Q} \) has the required properties.

To complete the proof observe that \( \mathcal{Q}^k \in \Phi^k(\varepsilon) \), \( \mathcal{Q}^k \succ \mathcal{R}^k \succ \mathcal{P} \), and

\[
|\mathcal{Q}^k| \leq |\mathcal{Q}|^k \leq \left( \frac{2}{\varepsilon} + 1 \right)^k 2^{pk^2}.
\]

\( \square \)

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References


