Degree powers in graphs: the Erdős-Stone theorem

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Dedicated to the memory of our dear friend Richard Schelp

Abstract

Let $1 \leq p \leq r + 1$, with $r \geq 2$ an integer, and let $G$ be a graph of order $n$. Let $d(v)$ denote
the degree of a vertex $v \in V(G)$. We show that if

$$\sum_{v \in V(G)} d^p(v) > (1 - 1/r)^p n^{p+1},$$

then $G$ has more than

$$\frac{1}{26r(r+1)r^r} n^{r-1}$$

$(r + 1)$-cliques sharing a common edge. From this we deduce that if

$$\sum_{v \in V(G)} d^p(v) > (1 - 1/r)^p n^{p+1} + C,$$

then $G$ contains more than

$$\frac{C}{p26r(r+1)+1r^r} n^{r-p}$$

cliques of order $r + 1$.

In turn, this statement is used to strengthen the Erdős–Stone theorem by using $\sum_{v \in V(G)} d^p(v)$
instead of the number of edges.

Keywords: joint; joint size; powers of degree; number of cliques; Turán graph; Erdős-Stone
theorem.

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‡Research supported in part by NSF grants DMS-0505550, CNS-0721983 and CCF-0728928, and ARO grant
W911NF-06-1-0076
§Research supported by NSF Grant # DMS-0906634
1 Introduction and main results

Given a graph $G$ and a vertex $u \in V (G)$, we write $d(u)$ for the degree of $u$. The sum

$$f_p (G) = \sum_{u \in V (G)} d^p (u),$$

is a much studied parameter in graph theory, especially for $p = 2$. In [5], Caro and Yuster raised a Turán type problem for $f_p (G)$: given $p \geq 1$, how large can $f_p (G)$ be if $G$ has no subgraph of a particular type. It turns out that such problems are usually more difficult for $p > 1$ than for $p = 1$, and often lead to involved analytical questions. Problems of this type have been subsequently studied by several authors (see e.g. [2],[6],[8],[11], and [12]). In particular, in [2] it was shown that for every real $p$, $1 < r < \frac{1}{p}$, if $G$ is a graph of sufficiently large order $n$ and has no clique of order $r + 1$, then

$$f_p (G) \leq f_p (T_r (n)),$$

where $T_r (n)$ denotes the $r$-partite Turán graph of order $n$.

In this note we continue this line of investigation and strengthen the Erdős–Stone theorem using $f_p (G)$ instead of the number of edges, that is to say, $f_1 (G) / 2$. This result is deduced from a number of other results about $f_p (G)$, which seem to be of some interest on their own.

We start with a general theorem about hereditary properties. Recall that a graph property is a class of graphs closed under graph isomorphisms. A graph property $Q$ is called hereditary if $G \in Q$ implies that every induced subgraph of $G$ also belongs to $Q$. As often, we set $Q_n = \{G \in Q \text{ and } V (G) = [n]\}$. We shall use both $f_p (G)$ and $\sum_{u \in V (G)} d^p (u)$ to improve readability.

**Theorem 1** Let $p \geq 1$ and let $Q$ be a hereditary property. Then the limit

$$\lim_{n \to \infty} \frac{1}{n^{p+1}} \max \left\{ \sum_{v \in V (G)} d^p (v) : G \in Q_n \right\}$$

exists.

In particular, for the Turán graph $T_r (n)$ one can show that

$$f_p (T_r (n)) \geq (1 - 1/r)^p n^{p+1} + O (n^{p-1}).$$

Thus, writing $K_r$ for the complete graph of order $r$, for the hereditary property of being $K_{r+1}$-free the limit in Theorem 1 is at least $(1 - 1/r)^p$. It turns out that for $p \leq r + 1$ there is also a matching upper bound.

**Theorem 2** Let $r \geq 2$ and $1 \leq p \leq r + 1$. If $G$ is a $K_{r+1}$-free graph of order $n$ then

$$\sum_{v \in V (G)} d^p (v) \leq (1 - 1/r)^p n^{p+1}. \quad (1)$$
We shall deduce Theorem 2 from the following analytical inequality.
If \( x_1, \ldots, x_r \) are nonnegative numbers such that \( x_1 + \cdots + x_r = 1 \), then
\[
\sum_{i=1}^{r} x_i (1 - x_i)^{r+1} < (1 - 1/r)^{r+1}
\]
unless \( x_1 = \cdots = x_r = 1/r \).

What Theorem 2 tells us is that if \( r \geq 2, 1 \leq p \leq r + 1 \), and \( G \) is a graph of order \( n \) with
\[
\sum_{v \in V(G)} d^p(v) > (1 - 1/r)^p n^{p+1}
\] 
then \( K_{r+1} \in G \). In fact, a stronger and subtler result holds. In order to state it, we need the following definitions from [2].

An \( r \)-joint of size \( t \) is a union of \( t \) distinct \( r \)-cliques sharing an edge. The maximum size of an \( r \)-joint in a graph \( G \) is called the \( r \)-joint size of \( G \) and is denoted by \( js_r(G) \).

The following theorem shows that condition (2) implies not only the existence of a single \( K_{r+1} \), but that of large \((r+1)\)-joints as well.

**Theorem 3** Let \( r \geq 2, 1 \leq p \leq r + 1 \), and let \( G \) be a graph of order \( n \). If
\[
\sum_{v \in V(G)} d^p(v) > (1 - 1/r)^p n^{p+1}
\] 
then \( js_{r+1}(G) > \frac{1}{2^{6r(r+1)r} n^{r-1}} \).

In the above theorem, the coefficient \( 2^{-6r(r-1)r} \) is far from being optimal, but it makes the statement valid for all \( n \); this coefficient can be increased when \( n \) is sufficiently large.

Joints play a crucial role in extremal and spectral graph theory (see, e.g., [9, 10]). A typical usage of them is in proving lower bounds on the number \( k_{r+1}(G) \) of \((r+1)\)-cliques, as in the following theorem.

**Theorem 4** Let \( r \geq 2, 1 \leq p \leq r + 1 \) and \( C > 0 \). If \( G \) is a graph of order \( n \) and
\[
\sum_{u \in V(G)} d^p(u) > (1 - 1/r)^p n^{p+1} + C,
\] 
then
\[
k_{r+1}(G) > \frac{C}{p^{2^{6r(r+1)+1}r} n^{r-p}}.
\]

Note that in this theorem \( C \) may depend on \( n \); in particular, we shall be especially interested in the case when it is in the range \( \Omega(n^p) \) to \( o(n^{p+1}) \). Denote by \( K_r(s_1, s_2, \ldots, s_r) \) the complete \( r \)-partite graph with class sizes \( s_1, s_2, \ldots, s_r \). Combining Theorem 4 with a result in [7], we shall immediately obtain the following strengthening of the classical Erdős–Stone theorem.
Theorem 5 Let \( r, p, c \) and \( n \) satisfy

\[
 r \geq 2, \quad 1 \leq p \leq r + 1, \quad \frac{c}{26^r(r+1)^1 + r^r} (r + 1) > (\log n)^{-1/(r+1)}.
\]

If \( G \) is a graph of order \( n \) and

\[
 \sum_{u \in V(G)} d^p (u) > (1 - 1/r)^p n^{p+1} + cn^{p+1},
\]

then \( G \) contains a \( K_{r+1} (s, \ldots, s, t) \), where

\[
 s = \left\lfloor \frac{c^{r+1}}{26^{r^3+1}r^2+7r^2-1} \frac{1}{(r+1)(r+1) \log n} \right\rfloor \quad \text{and} \quad t = \left\lceil n^{1-c^{-r}} \right\rceil. \quad (3)
\]

As in Theorem 4, the parameter \( c \) above may depend on \( n \), e.g., letting \( c = 1/\log \log n \), the conclusion is meaningful for sufficiently large \( n \). Observe also the following peculiarity of the graphs \( K_{r+1} (s, \ldots, s, t) \) in the conclusion of the theorem: if the statement holds for some \( c \), then it holds also for all positive \( c' < c \) provided \( n \) is large enough. That is to say, when \( n \) increases, in addition to the graph \( K_{r+1} (s, \ldots, s, t) \) guaranteed by the theorem, we can find other, larger and more lopsided graphs \( K_{r+1} (s', \ldots, s', t') \) with \( s' < s \) and \( t' > t \).

It is possible that a version of Theorem 5 can be deduced from some results of Pikhurko and Taraz [12] as well, but such a deduction does not seem too easy.

2 Proofs

Most of our notation is taken from [1]. Given a graph \( G \), we write:

- \( |G| \) for the number of vertices of \( G \);
- \( \Gamma_G (u) \) for the set of neighbors of a vertex \( u \), and \( d_G (u) \) for \( |\Gamma_G (u)| \);
- \( \delta (G) \) for the minimum degree of \( G \);
- \( G - u \) for the graph obtained by removing the vertex \( u \in V (G) \);
- \( G - uv \) for the graph obtained by removing the edge \( uv \in E (G) \);
- \( K_s (G) \) for the set of \( s \)-cliques of \( G \), and \( k_s (G) \) for \( |K_s (G)| \);
- \( K_r \) for the complete graph of order \( r \).

We shall need Lemma 5 of [3] and Theorem 1 of [7]: for ease of usage we restate them here.

**Lemma A** Let \( r \geq 2 \). If \( G \) is a graph of order \( n \), containing a \( K_{r+1} \), and with

\[
 \delta (G) > \left( 1 - \frac{1}{r} - \frac{1}{r^2 (r^2 - 1)} \right) n,
\]

then

\[
 j_{s_{r+1}} (G) > \frac{n^{r-1}}{r^{r+3}}.
\]
**Theorem B** Let \( r \geq 3 \), \((\log n)^{-1/r} \leq \alpha \leq 1/2\), and let \( G \) be a graph of order \( n \). If \( k_r(G) > \alpha n^r \), then \( G \) contains a \( K_r(s, \ldots, s, t) \) with \( s = [\alpha^r \log n] \) and \( t > n^{1 - r^{-1}} \).

In our proofs we shall use several versions of the inequality
\[
(1 + x)^p > 1 + px,
\]
which is valid for \( x \geq -1 \) and \( p \geq 1 \). In particular, one can easily deduce that if \( k > p - 1 \) and \( p \geq 1 \), then
\[
\left(1 + \frac{1}{k}\right)^p < 1 + \frac{p}{k + 1 - p}.
\]

(4)

**Proof of Theorem 1** Although our proof goes along familiar lines, we give it for the sake of completeness.

Set
\[
f_p(Q, n) = \max \{f_p(G) : G \in Q_n\},
\]
and let \( G \in Q_n \) be a graph such that \( f_p(Q, n) = f_p(G) \).

For every vertex \( u \in V(G) \) we have
\[
f_p(Q, n - 1) \geq \sum_{v \in V(G - u)} d_{G - u}^p(v) = \sum_{v \in V(G - u) \setminus \Gamma(u)} d_G^p(v) + \sum_{v \in \Gamma(u)} (d_G(v) - 1)^p \geq f_p(Q, n) - d_G^p(u) - p \sum_{v \in \Gamma(u)} d_G^{p - 1}(v).
\]

Summing this inequality for all \( u \in V(G) \), we obtain
\[
n f_p(Q, n - 1) \geq n f_p(Q, n) - \sum_{u \in V(G)} d_G^p(u) - p \sum_{u \in V(G)} \sum_{v \in \Gamma(u)} d_G^{p - 1}(v) = (n - p - 1) f_p(Q, n).
\]

When \( n > p + 1 \) we see that
\[
\frac{f_p(Q, n)}{f_p(Q, n - 1)} \leq \frac{n}{n - p - 1} = 1 + \frac{p + 1}{n - p - 1} < \left(1 + \frac{1}{n - p - 1}\right)^{p+1} \leq \frac{(n - p)^{p+1}}{(n - p - 1)^{p+1}}.
\]

Thus, for \( n > p \) the sequence
\[
\frac{f_p(Q, n)}{(n - p)^{p+1}}
\]
is nonincreasing, and therefore it is converging; clearly, so is \( f_p(Q, n) / n^{p+1} \) as well, completing the proof.

**Proof of Theorem 2** For \( 1 \leq p \leq r + 1 \), Jensen’s inequality implies that
\[
\frac{1}{n} \sum_{u \in V(G)} d^p(u) \leq \left( \frac{1}{n} \sum_{u \in V(G)} d^{r+1}(u) \right)^{p/(r+1)}
\]

5
so it suffices to prove inequality (1) for $p = r + 1$.

In [4] Erdős showed that if $G$ is a $K_{r+1}$-free graph, then there exists an $r$-partite graph $H$ with $V(H) = V(G)$ such that $d_G(u) \leq d_H(u)$ for every $u \in V(G)$. Clearly this implies that the maximum of $f_{r+1}(G)$ among the $K_{r+1}$-free graphs $G$ of order $n$ is attained on some complete $r$-partite graph, say the complete $r$-partite graph $K_r(n_1, \ldots, n_r)$ of order $n$ with part sizes $n_1, \ldots, n_r$. In this case we have

$$ f_{r+1}(K_r(n_1, \ldots, n_r)) = \sum_{i=1}^{r} n_i (n - n_i)^{r+1}, $$

and so setting

$$ F(x_1, \ldots, x_r) = \sum_{i=1}^{r} x_i (1 - x_i)^{r+1}, $$

we see that

$$ \frac{f_p(G)}{n^{r+2}} \leq \max \{ F(x_1, \ldots, x_r) : x_1 + \cdots + x_r = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, \ldots, r \}. \quad (6) $$

We shall prove that

$$ F(x_1, \ldots, x_r) \leq (1 - 1/r)^{r+1}, $$

with equality possible only if $x_1 = \cdots = x_r$. Let $(x_i)_1^r$ be a vector for which the maximum in (6) is attained and assume that $x_1 \leq x_2 \leq \cdots \leq x_r$.

Assume first that $r = 2$, and choose $x$ so that $x_1 = (1 - x)/2$, $x_2 = (1 + x)/2$, where $0 \leq x \leq 1$. Then

$$ F(x_1, x_2) = \frac{1 - x}{2} \left( 1 - \frac{1-x}{2} \right)^3 + \frac{1+x}{2} \left( 1 - \frac{1+x}{2} \right)^3 $$

$$ = \frac{1}{16} \left( (1-x)^2 + (1+x)^2 \right) \left( (1-x)^2 + (1+x)^2 \right) = \frac{1}{8} (1 - x^4). $$

Obviously $F(x_1, x_2)$ is maximum when $x = 0$, and so $x_1 = x_2$.

Let us now assume that $r \geq 3$. Routine calculations show that the function $x (1 - x)^{r+1}$ increases for $0 \leq x \leq 1/(r+2)$ and decreases for $1/(r+2) \leq x \leq 1$; also, it is concave for $0 \leq x \leq 2/(r+2)$ and convex for $2/(r+2) \leq x \leq 1$. If $x_r \leq 2/(r+2)$, the concavity of $x (1 - x)^{r+1}$ implies that $x_r = x_1$ and the proof is completed, so we shall assume that $x_r > 2/(r+2)$. Note that in fact $x_r < 1$, for otherwise $F(x_1, \ldots, x_r) = 0$. Thus, the convexity of $x (n-x)^{r+1}$ in the interval $(2/(r+2), 1)$ implies that only $x_r$ belongs to this interval. Again, the concavity of $x (1 - x)^{r+1}$ in the interval $[0, 2/(r+2)]$ implies that $x_1 = \cdots = x_{r-1}$. Hence, setting $x = x_1$, we see that $x_r = 1 - (r-1)x$.

Using Lagrange multipliers, we find that

$$ \frac{\partial F(x_1, \ldots, x_r)}{\partial x_1} = \frac{\partial F(x_1, \ldots, x_r)}{\partial x_r} $$

which in our case implies that

$$ ((r+2)x - 1)(1-x)^r = (r+1 - (r+2)(r-1)x)(r-1)^r x^r. \quad (7) $$
We shall show that equality (7) is not possible. Assume for a contradiction that (7) holds for some \( x \). Clearly we have \( x > 1/(r+2) \) and from \( 1 - (r-1)x > 2/(r+2) \), we see that

\[
\frac{1}{r+2} < x < \frac{r}{(r+2)(r-1)}.
\]

Write \( I \) for the open interval \((1/(r+2), r/((r+2)(r-1))]\) and note first that (7) is equivalent to

\[
-r + 1 + \frac{2}{(r+2)x-1} = \frac{1}{(r-1)^r} \left( \frac{1}{x} - 1 \right)^r. \tag{8}
\]

Let

\[
g(x) = -r + 1 + \frac{2}{(r+2)x-1},
\]
\[
h(x) = \frac{1}{(r-1)^r} \left( \frac{1}{x} - 1 \right)^r,
\]

and let \( L(x) \) be the linear function

\[
L(x) = r - 1 - 2(r+2)(r-1)^2 \left( x - \frac{r}{(r+2)(r-1)} \right) = g \left( \frac{r}{(r+2)(r-1)} \right) + \left( x - \frac{r}{(r+2)(r-1)} \right) g' \left( \frac{r}{(r+2)(r-1)} \right).
\]

Note that both \( g(x) \) and \( h(x) \) are decreasing convex functions in \( I \). Hence, by Taylor’s expansion, \( g(x) \geq L(x) \) for all \( x \in I \). We shall show that if \( r \geq 4 \) and \( x \in I \), then \( h(x) < L(x) \), which contradicts (8) and completes the proof for \( r > 3 \). Indeed, if \( r \geq 4 \), we find that

\[
h \left( \frac{1}{r+2} \right) = \frac{(r+1)^r}{(r-1)^r} < 3r - 3 = L \left( \frac{1}{r+2} \right),
\]

and for \( r \geq 3 \), we find that

\[
h \left( \frac{r}{(r+2)(r-1)} \right) = \frac{(r^2 - 2)^r}{r^r(r-1)^r} < r - 1 = L \left( \frac{r}{(r+2)(r-1)} \right).
\]

For \( r = 3 \) we can apply the same method with a slight modification. Note that we have

\[
h(x) < h \left( \frac{1}{5} \right) = 8,
\]

while for \( x > 6/25 \),

\[
g(x) > g(6/25) = 8.
\]

This means that if \( x \) is a solution to (8), then \( x > 6/25 \). To finish the proof note that

\[
h(6/25) = \frac{6859}{1728} < \frac{22}{5} = 14 - 40 \cdot \frac{6}{25} = L \left( \frac{6}{25} \right).
\]
Therefore equation (8) has no solution in \( I \), and the proof is completed. \( \square \)

**Proof of Theorem 4** We shall deduce Theorem 4 from Theorem 3. Since \( f_p(G) > (1 - 1/r)^p n^{p+1} \), Theorem 3 implies that \( j_s r+1(G) > n^{r-1} / (2^{6r(r+1)r}) \), which means that some edge of \( G \) is contained in at least \( n^{r-1} / (2^{6r(r+1)r}) \) cliques of order \( r + 1 \). Set \( k = 0 \) and \( G_0 = G \), and perform the following procedure:

**repeat**

- find an edge \( uv \) in \( G_k \) that is contained in at least \( n^{r-1} / (2^{6r(r+1)r}) \) cliques of order \( r + 1 \);
- let \( G_{k+1} = G_k - uv \);
- add 1 to \( k \);

until \( f_p(G_k) \leq (1 - 1/r)^p n^{p+1} \).

Note that at the beginning of each iteration we have \( f_p(G_k) > (1 - 1/r)^p n^{p+1} \), and by Theorem 3, \( j_s r+1(G_k) > n^{r-1} / (2^{6r(r+1)r}) \), i.e., there is an edge \( uv \) contained in at least \( n^{r-1} / (2^{6r(r+1)r}) \) cliques of order \( r + 1 \). To estimate how the removal of \( uv \) affects \( f_p(G_k) \), we find that

\[
\sum_{w \in V(G)} d_{G_{k+1}}^p(w) = \sum_{w \in V(G) \setminus \{u,v\}} d_{G_k}^p(w) + (d_{G_k}(v) - 1)^p + (d_{G_k}(u) - 1)^p
\]

\[
\geq \sum_{w \in V(G)} d_{G_k}^p(w) - pd_{G_k}^{p-1}(v) - pd_{G_k}^{p-1}(u) > \sum_{w \in V(G)} d_{G_k}^p(w) - 2pn^{p-1}.
\]

Therefore, upon exiting the procedure, we have

\[
(1 - 1/r)^p n^{p+1} \geq f_p(G_k) = \sum_{w \in V(G)} d_{G_k}^p(w) > \sum_{w \in V(G)} d_{G_k}^p(w) - 2kn^{p-1}
\]

\[
> (1 - 1/r)^p n^{p+1} + C - 2kn^{p-1}
\]

and so,

\[
k > \frac{C}{2pn^{p-1}}.
\]

Hence, by removing edges the procedure destroys at least

\[
k \frac{1}{2^{6r(r+1)r}} n^{r-1} > \frac{C}{2p2^{6r(r+1)r}} n^{r-p}
\]

\((r + 1)\)-cliques, implying the assertion. \( \square \)

To simplify the proof of Theorem 3, we give two preliminary lemmas. Although the second one seems just a general version of the first, a number of different details require two separate lemmas.

**Lemma 6** Let \( G \) be a graph of order \( n \), containing a \( K_3 \). If \( j_s 3(G) \leq n/1500 \) and \( u \) is a vertex with \( d_G(u) = \delta(G) \), then

\[
\sum_{v \in V(G - u)} d_{G-u}^3(v) > \sum_{v \in V(G)} d_G^3(v) - \frac{499}{1000} n^3.
\]
Proof Since \( js_3 (G) \leq n/1500 < n/32 \), Lemma A implies that \( \delta (G) \leq 5n/12 \). Let \( u \) be a vertex such that \( d (u) = \delta (G) \), and note that

\[
\sum_{v \in V(G-u)} d_{G-u}^3 (v) = \sum_{v \in V(G-u) \setminus \Gamma_G (u)} d_G^3 (v) + \sum_{v \in \Gamma_G (u)} (d_G (v) - 1)^3
\]

\[
> \sum_{v \in V(G)} d_G^3 (v) - \delta^3 - 3 \sum_{v \in \Gamma_G (u)} d_G^2 (v) .
\]

In particular, if \( v \in \Gamma_G (u) \), then \( |\Gamma_G (v) \cap \Gamma_G (u)| \leq js_3 (G) \leq n/1500 \), and therefore

\[
d_G (v) + \delta \leq n + n/1500.
\]

Hence,

\[
-3 \sum_{v \in \Gamma_G (u)} d_G^2 (v) \geq -3 \sum_{v \in \Gamma_G (u)} (n + n/1500 - \delta)^2 = -3\delta (n + n/1500 - \delta)^2,
\]

implying that

\[
\sum_{v \in V(G-u)} d_{G-u}^3 (v) > \sum_{v \in V(G)} d_G^3 (v) - 3\delta (n + n/1500 - \delta)^2 - \delta^3.
\]

Simple calculations show that \(-3\delta (n + n/1500 - \delta)^2 - \delta^3\) is nonincreasing in \( \delta \), and so

\[
\sum_{v \in V(G-u)} d_{G-u}^3 (v) > \sum_{v \in V(G)} d_G^3 (v) - 3(5/12) (1 + 1/1500 - 5/12)^2 n^3 - (5/12)^3 n^3
\]

\[
> \sum_{v \in V(G)} d_G^3 (v) - \left( \frac{1}{2} - \frac{1}{1000} \right) n^3,
\]

completing the proof. \( \Box \)

We find that proving Lemma 7 before the proof of Theorem 3 greatly simplifies the latter. However, the logical position of Lemma 7 in our argument needs clarification as it may seem circular reasoning. We use Lemma 7 to prove Theorem 3 by induction on \( r \), but to prove the lemma we apply Theorem 4 for \( r' < r \), which in turn is deduced from Theorem 3 for \( r' < r \). Therefore, the induction assumption makes our argument logically correct.

Lemma 7 Let \( r \geq 3 \) and let \( G \) be a graph of order \( n \), containing a \( K_{r+1} \). If

\[
js_{r+1} (G) \leq \frac{1}{2r^2 - r} n^{r-1}, \tag{9}
\]

and \( u \) is a vertex with \( d_G (u) = \delta (G) \), then

\[
\sum_{v \in V(G-u)} d_{G-u}^{r+1} (v) > \sum_{v \in V(G)} d_G^{r+1} (v) - \left( r + 2 - \frac{1}{2r^2 (r+1)} \right) (1 - 1/r)^{r+1} n^{r+1}.
\]
Proof Note that Lemma A, together with (9), implies that
\[
\delta (G) \leq \left( 1 - \frac{1}{r} - \frac{1}{r^2 (r^2 - 1)} \right) n, \tag{10}
\]

Set \( \delta = \delta (G) \) and let \( u \) be a vertex with \( d (u) = \delta \); note that
\[
\sum_{v \in V(G-u)} d^r_{G-u} (v) = \sum_{v \in V(G-u) \setminus \Gamma_G (u)} d^r_{G} (v) + \sum_{v \in \Gamma_G (u)} (d_G (v) - 1)^{r+1} \geq \sum_{v \in \Gamma_G (u)} d^r_{G} (v) - \delta^{r+1} - (r + 1) \sum_{v \in \Gamma_G (u)} d^r_{G} (v). \tag{11}
\]

It is easy to see that the assertion holds if \( \delta \leq n/8 \). Indeed, in view of (11), all we need to show in this case is that
\[
\delta^{r+1} + (r + 1) \sum_{v \in \Gamma_G (u)} d^r_{G} (v) < \left( r + 2 - \frac{1}{2r^2 (r + 1)} \right) (1 - 1/r)^{r+1} n^{r+1}. \]

This inequality does hold, since by (10) and in view of \((1 - 1/r)^{r+1} \geq 1/8\), we find that
\[
\delta^{r+1} + (r + 1) \sum_{v \in \Gamma_G (u)} d^r_{G} (v) < \frac{n^{r+1} + (r + 1) \delta n^r}{8^{r+1}} \leq \frac{1}{8^r} (1 - 1/r)^{r+1} n^{r+1} + (r + 1) \frac{n^{r+1}}{8} \leq \left( 1 - \frac{1}{2r^2 (r + 1)} \right) \left( 1 - 1/r \right)^{r+1} n^{r+1} + (r + 1) (1 - 1/r)^{r+1} n^{r+1}.
\]

Therefore, to the end of the proof we may and shall assume that \( \delta > n/8 \).

Set \( H = G [\Gamma_G (u)] \). Note first that \( k_r (H) \leq \frac{1}{2^{6r^2} r^{r+1}} n^r \), \( \tag{12} \)

for otherwise there would be a vertex in \( \Gamma (u) \) that is common to at least
\[
\frac{rk_r (H)}{\delta} > \frac{n^{r-1}}{2^{6r^2} r^r}
\]
cliques in \( K_r (H) \), and so
\[
\mathcal{J}_{s+1} (G) > \frac{1}{2^{6r^2} r^r} n^{r-1},
\]
contradicting (9).

We shall use inequality (12) to prove that for every \( k = 1, \ldots, r \),
\[
\sum_{v \in \Gamma_G (u)} d^k_{H} (v) \leq \left( 1 - \frac{1}{r-1} \right)^k \delta^{k+1} + \frac{2 (r - 1)^{r-1}}{8^{r+1} r^r} \delta n^k. \tag{13}
\]
Indeed, if this inequality fails for some \( k \in [r] \), then applying Theorem 4 to the graph \( H \) with \( p = k \) and
\[
C = \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} \delta n^k,
\]
we find that
\[
k_r(H) > \frac{1}{k \cdot 2^{6r(r-1)+1} (r-1)^{r-1}} \cdot \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} \delta n^k \delta^{r-1-k} > \frac{1}{2^{6r^2-3r+3}r^{r+1}} \cdot \frac{n^r}{8^{r-k}} \geq \frac{1}{2^{6r^2}r^{r+1}} n^r,
\]
contradicting (12).

Now, using inequality (13) and the inequality \( d_G(v) \leq n - \delta + d_H(v) \) whenever \( v \in \Gamma_G(u) \), we obtain
\[
\sum_{v \in \Gamma(u)} d_G^r(v) \leq \sum_{v \in \Gamma(u)} (n - \delta + d_H(v))^r = \sum_{i=0}^{r} \binom{r}{i} (n - \delta)^{r-i} \sum_{v \in \Gamma(u)} d_H^i(v)
\]
\[
\leq \delta (n - \delta)^r + \delta \sum_{i=1}^{r} \binom{r}{i} (n - \delta)^{r-i} \left( \left( 1 - \frac{1}{r-1} \right)^i \delta^i + \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} n^i \right)
\]
\[
= \delta \sum_{i=0}^{r} \binom{r}{i} (n - \delta)^{r-i} \left( 1 - \frac{1}{r-1} \right)^i \delta^i + \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} \delta \sum_{i=1}^{r} \binom{r}{i} (n - \delta)^{r-i} n^i
\]
\[
< \delta \left( n - \frac{\delta}{r-1} \right)^r + \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} \delta (2n - \delta)^r.
\]
\[
< \delta \left( n - \frac{\delta}{r-1} \right)^r + \frac{2(r - 1)^{r-1}}{8^{r+1}r^r} 2^r \left( 1 - \frac{1}{r} \right) n^{r+1}
\]
\[
= \delta \left( n - \frac{\delta}{r-1} \right)^r + \frac{1}{4^{r+1}} \frac{(r-1)^r}{r^{r+1}} n^{r+1}.
\]

A simple proof by induction shows that
\[
\frac{1}{4^{r+1}} \frac{(r-1)^r}{r^{r+1}} < \frac{1}{2^{r^2} (r+1)^2} \left( 1 - \frac{1}{r} \right)^{r+1}
\]
for \( r \geq 3 \), and so, in view of (11), we get
\[
\sum_{v \in V(G-u)} d_{G-u}^{r+1}(v) > \sum_{v \in V(G)} d_G^{r+1}(v) - \delta^{r+1} - (r+1) \delta \left( n - \frac{\delta}{r-1} \right)^r - \frac{1}{2^{r^2} (r+1)} \left( 1 - \frac{1}{r} \right)^{r+1} n^{r+1}.
\]

On the other hand, using calculus, one can show that the function
\[
\delta^{r+1} + (r+1) \delta \left( n - \frac{\delta}{r-1} \right)^r
\]

(14)
is increasing in $\delta$, and so, recalling (10), we find that

$$\delta^{r+1} + (r + 1) \delta \left( n - \frac{\delta}{r - 1} \right)^r \leq \left( 1 - \frac{1}{r^2} - \frac{1}{(r - 1)^2} \right)^{r+1}$$

$$+ (r + 1) \left( \frac{1}{r} - \frac{1}{r^2 (r - 1)} \right) \left( 1 - \frac{1}{r - 1} \left( 1 - \frac{1}{r} - \frac{1}{r^2 (r - 1)} \right) \right)^r$$

$$= \left( 1 - \frac{1}{r} - \frac{1}{r^2 (r - 1)} \right)^{r+1}$$

$$+ (r + 1) \left( \frac{1}{r} - \frac{1}{r^2 (r - 1)} \right) \left( 1 - \frac{1}{r} + \frac{1}{r^2 (r - 1)^2 (r + 1)} \right)^r.$$ 

To estimate the right-hand side of this inequality, let us divide both sides by $(1 - 1/r)^{r+1}$, thus obtaining

$$\left( \delta^{r+1} + (r + 1) \delta \left( n - \frac{\delta}{r - 1} \right)^r \right) (1 - 1/r)^{-r-1} \leq \left( 1 - \frac{1}{r} - \frac{1}{r (r - 1)^2 (r + 1)} \right)^{r+1}$$

$$+ (r + 1) \left( \frac{1}{r} - \frac{1}{r (r - 1)^2 (r + 1)} \right) \left( 1 - \frac{1}{r} + \frac{1}{r (r - 1)^2 (r + 1)} \right)^r.$$ 

We shall estimate the two terms of the right-hand side above separately. Using inequality (4) and some algebra, we see that

$$\left( 1 - \frac{1}{r} - \frac{1}{r (r - 1)^2 (r + 1)} \right)^{r+1} < 1 - \frac{r + 1}{r (r - 1)^2 (r + 1) + r + 1} = 1 - \frac{1}{r (r - 1)^2 + 1},$$

and also

$$\left( 1 + \frac{1}{r (r - 1)^3 (r + 1)} \right)^r < 1 + \frac{r}{r (r - 1)^3 (r + 1) - r} = 1 + \frac{1}{(r - 1)^3 (r + 1) - 1}.$$ 

Furthermore, from the last inequality we obtain

$$(r + 1) \left( 1 - \frac{1}{r (r - 1)^2 (r + 1)} \right) \left( 1 + \frac{1}{r (r - 1)^3 (r + 1)} \right)^r$$

$$< (r + 1) \left( 1 - \frac{1}{(r - 1)^3 (r + 1)} \right) \left( 1 + \frac{1}{(r - 1)^3 (r + 1) - 1} \right)$$

$$= (r + 1) \left( 1 + \frac{1}{(r - 1)^3 (r + 1) - 1} \right) - \frac{(r - 1)(r + 1)}{r ((r - 1)^3 (r + 1) - 1)}$$

$$\leq r + 1 + \frac{r + 1}{r ((r - 1)^3 (r + 1) - 1)}$$

$$\leq r + 1 + \frac{1}{r (r - 1)^3 - 1}.$$
Now, using the bounds for the two terms on the right-hand side of (15), we see that,

\[
\left( \delta^{r+1} + (r + 1) \delta \left( n - \frac{\delta}{r-1} \right)^r \right) (1 - 1/r)^{-r-1} < \left( r + 2 - \frac{1}{r(r-1)^2 + 1} + \frac{1}{r(r-1)^3 - 1} \right) n^{r+1} \\
< \left( r + 2 - \frac{1}{r^2(r+1)^2} \right) n^{r+1},
\]

which gives

\[
\delta^{r+1} + (r + 1) \delta \left( n - \frac{\delta}{r-1} \right)^r < \left( r + 2 - \frac{1}{r^2(r+1)^2} \right) (1 - 1/r)^{r+1} n^{r+1}.
\]

Hence, from (14) we obtain

\[
\sum_{v \in V(G)} d^{r+1}_{G-u}(v) > \sum_{v \in V(G)} d^{r+1}_G(v) - \left( r + 2 - \frac{1}{2r^2(r+1)^2} \right) (1 - 1/r)^{r+1} n^{r+1},
\]

completing the proof. \(\square\)

**Proof of Theorem 3** We shall apply induction on \(r\), making use of the fact that if Theorem 3 holds for some \(r\), then Theorem 4 and Lemma 7 hold for the same \(r\) as well.

Let us first prove the assertion for \(r = 2\). We claim that in this case

\[
\text{js}_3(G) > n/16000 > \frac{1}{238} n,
\]

as required. Assume for a contradiction that \(\text{js}_3(G) \leq n/16000\). Theorem 2 implies that \(K_3 \subseteq G\), and so

\[
n \geq 16000 \text{js}_3(G) \geq 16000.
\]

Set \(i = 0\) and \(G_0 = G\), and let us perform the following procedure:

**while** \(\text{js}_3(G_i) \leq |G_i|/1500\) **do begin**

select a vertex in \(G_i\) such that \(d_{G_i}(u) = \delta(G_i)\);

**let** \(G_{i+1} = G_i - u\);

**add** 1 to \(i\);

**end**

Note that

\[
\frac{1}{8} k^4 - \frac{499}{1000} k^3 > \frac{1}{8} (k - 1)^4,
\]

for \(k > 1500\). Hence, we see that if \(|G_i| \geq n/10 > 1500\) and \(f_3(G_i) > |G_i|^4/8\) at the beginning of the while loop, then \(G_i\) contains a \(K_3\), and since \(\text{js}_3(G_i) \leq |G_i|/1500\), Lemma 6 implies that \(G_{i+1}\) also satisfies \(f_3(G_{i+1}) > |G_{i+1}|^4/8\).
We claim that at the beginning of the while loop we always have \( |G_i| > n/10 \). Indeed, otherwise at some iteration we would have \( |G_i| = \lceil n/10 \rceil \), and the following inequalities would hold
\[
[n/10]^4 = |G_i|^4 > f_3(G_i) > f_3(G_1) - \frac{499}{1000} \sum_{i=1}^{n} \epsilon^3 > \frac{1}{8} n^4 - \frac{499}{1000} (n+1)^4.
\]

Hence,
\[
\frac{n^4}{10^4} > \frac{1}{8} n^4 - \frac{499}{4000} (n+1)^4,
\]
and so
\[
1 \mu(G) - \mu(G) \frac{3}{2498} = \frac{499}{4000} \left( \frac{1}{8} - \frac{1}{10^4} \right) > \frac{n^4}{(n+1)^4} > 1 - \frac{4}{n+1} > 1 - \frac{4}{16000}.
\]

This contradiction shows that, when the procedure stops, we have \( |G_i| > n/10 \). Now since
\[
j_3(G_i) > 1500 |G_i|,
\]
we obtain
\[
j_3(G) \geq j_3(G_i) > \frac{1}{1500} |G_i| > \frac{1}{16000} n,
\]
as claimed.

Assume now that \( r \geq 3 \) and that the assertion holds for all integers between 2 and \( r-1 \). Assume for a contradiction that
\[
j_{s+1}(G) \leq \frac{1}{26^r(r+1)^2} n^{r-1}.
\]
Since Theorem 2 implies that \( K_{r+1} \subset G \), we see that
\[
n \geq \left( 2^{6(r+1)} r^r j_{s+1}(G) \right)^{1/(r-1)} \geq \left( 2^{6(r+1)} r^r \right)^{1/(r-1)} > 8^{r+2} r.
\]
Furthermore, as in the case \( r = 2 \), we set \( i = 0 \), \( G_0 = G \), and perform the following procedure:

\textbf{while} \( j_{s+1}(G) \leq 2^{-6r^2 r} |G_i|^{r-1} \) \textbf{do begin}

select a vertex in \( G_i \) such that \( d_{G_i}(u) = \delta(G_i) \);
let \( G_{i+1} = G_i - u \);
add 1 to \( i \);
\textbf{end}

Note that if a number \( k \) satisfies \( k > 2r^2 (r+1) (r+2)^2 \), then
\[
(1 - 1/r)^{r+1} k^{r+2} - \left( r + 2 - \frac{1}{2 r^2 (r+1)} \right) (1 - 1/r)^{r+1} k^{r+1} > (1 - 1/r)^{r+1} (k-1)^{r+2}.
\]
Hence, if \( |G_i| \geq n/8 \) and \( f_{r+1}(G_i) > (1 - 1/r)^{r+1} |G_i|^{r+2} \) at the beginning of the while loop, then \( G_i \) contains a \( K_{r+1} \) and since \( j_{s+1}(G) \leq 2^{-6r^2 r} |G_i|^{r-1} \), Lemma 7 implies that \( G_{i+1} \) also satisfies \( f_{r+1}(G_{i+1}) > (1 - 1/r)^{r+1} |G_{i+1}|^{r+2} \), in view of
\[
|G_i| \geq n/8 > 8^{r+1} r > 2r^2 (r+1) (r+2)^2.
\]
We claim that at the beginning of the \textbf{while} loop we always have $|G_i| > n/8$. Indeed, otherwise at some iteration we would have $|G_i| = [n/8]$, and the following inequalities would hold

$$\left(\frac{n}{8}\right)^{r+2} > f_{r+1}(G_i) > f_{r+1}(G_1) - \left(r + 2 - \frac{1}{2r^2 (r+1)}\right) \left(1 - 1/r\right)^{r+1} \sum_{i=1}^{n} i^{r+1}$$

$$> (1 - 1/r)^{r+1} n^{r+2} - \left(r + 2 - \frac{1}{2r^2 (r+1)}\right) \left(1 - 1/r\right)^{r+1} \frac{(n+1)^{r+2}}{r+2},$$

implying that

$$\left(r + 2 - \frac{1}{2r^2 (r+1)}\right) (1 - 1/r)^{r+1} \frac{(n+1)^{r+2}}{r+2} > ((1 - 1/r)^{r+1} - 1/8^{r+2}) n^{r+2},$$

and so,

$$1 - \frac{1}{2r^2 (r+1) (r+2)} > \left(1 - \frac{1}{8^{r+2} (1 - 1/r)^{r+1}}\right) \frac{n^{r+2}}{(n+1)^{r+2}}$$

$$> \left(1 - \frac{1}{8^{r+2} (1 - 1/r)^{r+1}}\right) \left(1 - \frac{r+2}{n+1}\right)$$

$$> 1 - \frac{1}{8^{r+2} (1 - 1/r)^{r+1}} - \frac{r+2}{n+1}$$

$$> 1 - \frac{1}{8^{r+2} r} - \frac{r+2}{8^{r+2} r}$$

It is easy to see that for $r \geq 3$ we have

$$\frac{1}{2r^2 (r+1) (r+2)} > \frac{1}{8^{r+1}} + \frac{r+2}{8^{r+2} r},$$

and this contradiction shows that, when the procedure stops, we have $|G_i| > n/8$.

Therefore,

$$j_{s_{r+1}}(G) \geq j_{s_{r+1}}(G_i) > \frac{1}{2^{6r^2 r} r} \frac{n^{r-1}}{8^{r-1}} > \frac{1}{2^{6r(r+1)r} r} \frac{n^{r-1}}{r+1}.$$ 

This completes the induction step and the proof of the theorem. \hfill \Box

\textbf{Proof of Theorem 5} The proof is an immediate consequence of Theorem 4 and Theorem B. Indeed, setting

$$C = cn^{p+1},$$

Theorem 4 implies that

$$k_{r+1}(G) > \frac{C}{p2^{6r(r+1)+1} r} n^{r-p} > \frac{c}{2^{6r(r+1)+1} r (r+1)} n^{r+1}.$$
Now letting
\[ \alpha = \frac{c}{2^{6r(r+1)+1}r^r(r+1)}, \]
we see that \( \alpha > (\log n)^{-1/(r+1)} \) and so Theorem B implies that \( G \) contains a \( K_{r+1}(s', \ldots, s', t') \), where
\[ s' = [\alpha^{r+1} \log n] \quad \text{and} \quad t' > n^{1-\alpha^r}. \]
Note that here \( s' \) is equal to \( s \) in (3) and \( t' \) is greater than or equal to \( t \) in (3), so Theorem 5 is proved.

3 Concluding remarks

Here we formulate several open questions related the general topic of this paper:

Given an integer \( r \geq 2 \), what is the maximum \( p = p(r) \) such that if \( G \) is a \( K_{r+1} \)-free graph, then \( f_p(G) \leq f_p(T_r(|G|)) \) for every graph \( G \)?

Another, formally similar, but otherwise rather different, question was raised in [2] and is reiterated below:

Given an integer \( r \geq 2 \), what is the maximum \( p = p(r) \) such that if \( G \) is a \( K_{r+1} \)-free graph, then \( f_p(G) \leq f_p(T_r(|G|)) \) provided \( |G| \) is sufficiently large?

In [2], we have determined that \( p_2 = 3 \) and in general \( p_r < r + \sqrt{2r} \). Given the result of Erdős [4] used in the proof of Theorem 2, this is an essentially analytical and number-theoretical problem. However, contrary to the authors of [12], we believe that finding \( p_r \) is important, as it may have practical implications.

In fact, in [12], Pikhurko and Taraz discuss even more general problems, namely, letting \( \varphi \) be a nonnegative increasing real function and setting
\[ f_\varphi(G) = \sum_{u \in V(G)} \varphi(d_G(u)), \]
they raise Turán type and Erdős-Stone type extremal questions using \( f_\varphi(G) \) instead of the number of edges. Under certain restriction on \( \varphi \), they give several nice results in a fairly general setup. Nonetheless, we believe that for possible applications it is important to study concrete functions \( \varphi \), for which the results of [12] say nothing. We end up with one such example:

Given an integer \( r \geq 2 \), what is the maximum integer \( k \) such that if \( G \) is a \( K_{r+1} \)-free graph of order \( n \), then
\[ \sum_{u \in V(G)} \binom{d_G(u)}{k} \leq \sum_{u \in V(T_r(n))} \binom{d_{T_r(n)}(u)}{k}. \]

The combinatorial implications of this question are obvious as
\[
\sum_{u \in V(G)} \binom{d_G(u)}{k}
\]
counts the \((k + 1)\)-vertex subgraphs of \(G\) with a dominating vertex.

**Acknowledgement**
The authors are indebted to Paul Balister for his helpful remarks and suggestions.

**References**


