Nearly bipartite graphs

E. Győri¹,², V. Nikiforov¹, R. H. Schelp¹

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Abstract

We prove that if a nonbipartite graph $G$ on $n$ vertices has minimal degree

$$\delta \geq \frac{n}{4k+2} + c_{k,m}$$

where $c_{k,m}$ does not depend on $n$ and $n$ is sufficiently large, if $C_{2s+1} \subset G$ for some $k \leq s \leq 4k+1$ then $C_{2s+2j+1} \subset G$ for every $j = 1, \ldots, m$.

We give a structural description of all graphs on $n$ vertices with

$$\delta \geq \frac{n}{4k + 2}$$

and not containing odd cycles of order larger than $2k+1$ and show that they can be made bipartite by deletion of a fixed number of edges or vertices. Such graphs will be called nearly bipartite graphs.

Keywords: odd cycle lengths, minimal degree, nearly bipartite graphs

1 Introduction

Throughout the paper standard notation is used as found in [2]. It is assumed that all graphs are defined on the vertex set $[n] = \{1, 2, \ldots, n\}$ and for any vertex $i$, $N_i$ is the set of its neighbors.

A classical question in graph theory is how little must a graph be altered to make it bipartite or which properties make the graph nearly bipartite. It is well known that, in general, almost one-half of the edges of a graph may need to be deleted to make it bipartite, while a long standing conjecture of Erdős [6] says that a triangle-free graph of order $n$ can be made bipartite by deleting at most $n^2/25$ of its edges. In [7], a conjecture of Erdős and Sós was proved showing that there are functions $f(c)$ and $g(c)$ such that if $G$ is a graph on $n$ vertices with no odd cycle of length less than $cn$, then $G$ can be made bipartite by deleting either $f(c)$ vertices or $g(c)$ edges.

¹ Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee, 38152.

² On leave from Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary.

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We call a nonbipartite graph *nearly bipartite* if it can be made bipartite by the deletion of a fixed number of its vertices or edges, independent of the order of the graph. Thus in the result from [7] mentioned in the previous paragraph, the graph $G$ is nearly bipartite. Clearly "blown-up" odd cycles $C_{2k+1}$ of order $n$ with minimum degree $\delta = \frac{2n}{2k+1}$, $n$ large and $k$ fixed, are not nearly bipartite. It is reasonable to consider nonbipartite graphs of order $n$ with $\delta > \frac{2n}{2k+1}$.

Problems of this type have been studied with representative results given in the next three theorems.

**Theorem 1.** [1] Every nonbipartite graph $G$ on $n$ vertices with $\delta(G) > \frac{2n}{2k+1}$ contains an odd cycle of length at most $2k - 1$. □

**Theorem 2.** [8] Let $G$ be a nonbipartite graph on $n$ vertices with $n > \binom{k+1}{2}(2k+1)(3k-1)$ and $\delta(G) > \frac{2n}{2k+1}$. Then, either $G$ contains a $C_{2k-1}$ or $G$ contains no odd cycle of length greater than $k/2$. □

**Theorem 3.** [4] Let $k \geq 3$ be a fixed odd positive integer. If $G$ is a 2-connected nonbipartite graph on $n$ vertices with $\delta(G) \geq \frac{2n}{k+2}$, then for $n > n_0(k)$ either $G$ contains the $k$-cycle $C_k$ or $n$ is a multiple of $k+2$ and $G$ is isomorphic to the blown-up $C_{k+2}$. □

From Theorem 2, if $G$ is nonbipartite of order $n$ with $\delta > \frac{2n}{8k+5}$ and no $C_{8k+3}$, then for large $n$, $G$ has only odd cycles of length at most $2k + 1$. We show in Theorem 10 that under these conditions $G$ is nearly bipartite; in fact, in Theorem 10 and what follows we give a structural description of such graphs.

Many authors (e.g., see [3] and [9]) have studied intervals of consecutive even cycle lengths in graphs with large minimal degree. In this vein, we show in Theorem 7 that under some given conditions the set of all cycle lengths of a graph $G$ with large minimal degree contains arbitrarily large intervals of consecutive odd numbers. Consequently, Theorem 7 and Theorem 10 imply that if $G$ is of order $n$ and is not nearly bipartite, and $\delta(G) > cn$ for some $c > 0$, then the set of all cycle lengths of $G$ contains arbitrarily large intervals of consecutive odd numbers.

### 2 Main results

We start by recalling a well-known theorem of Erdős and Gallai [5] giving a relation between the size of a graph and the order of its maximal path.
**Theorem 4.** Suppose \( k \geq 2 \) is an integer. If \( G \) is a graph on \( n \) vertices and more than \( kn \) edges then \( G \) contains a \((2k+2)\)-path. In particular, if \( \delta(G) > 2k \) then \( G \) contains a path on \( 2k+2 \) vertices. \( \square \)

It is easy to derive the following simple corollary.

**Corollary 5.** Suppose \( k \geq 1 \) is an integer and \( G \) is a bipartite graph with parts \( A \) and \( B \). If \( G \) has more than \( k(|A| + |B|) \) edges, then for every \( j \in [k] \), it contains a path on \( 2j+1 \) vertices with both endvertices in \( A \). \( \square \)

The following lemma is needed in the proof of Theorem 7.

**Lemma 6.** Suppose \( c \) is a real number with \( 0 < c < 1 \) and \( r \geq 2 \), \( l \geq 0 \) are integers. Let \( G \) be a graph on \( n \) vertices and minimum degree \( \delta \geq cn \). If \( n \) is large enough and for some \( r \)-path \( u, \ldots, v \),

\[
|N_u \cap N_v| \geq \frac{2l}{c} + r
\]

then \( G \) contains a cycle \( C_{r+2j+1} \) for every \( j = 0, \ldots, l \).

**Proof.** Let

\[
n \geq \frac{2}{c} \left( \frac{2l}{c} + r \right)
\]

Assume, wlog, \( 1, 2, \ldots, r \) is an \( r \)-path \( P \) with \( |N_1 \cap N_r| \geq 2l/c + r \). Set \( q = \lceil 2l/c \rceil \) and select a set \( A \subset N_1 \cap N_r \) with \( |A| = q \) so that \( A \) has no vertices in common with \( P \). Denote by \( B \) the set of all vertices which do not belong to \( A \) or \( P \). Clearly, \( |B| = n - q - r \). For the bipartite graph with parts \( A \) and \( B \) induced in \( G \) we have

\[
e(A, B) = \sum_{i \in A} |N_i \cap B| \geq q (\delta - q + 1 - r) \geq q \left( cn - \frac{2l}{c} - r \right) \\
\geq q \left( cn - \frac{cn}{2} \right) = \frac{qcn}{2} \geq nl > (n - r)l.
\]

(1)

For any vertex \( v \in A \) the sequence \( 1, 2, \ldots, r, v, 1 \) is an \((r+1)\)-cycle in \( G \), so the assertion is proved for \( l = 0 \). Fix some \( j \in [l] \). From Corollary 5, there is a \((2j+1)\)-path \( Q \) with both end-vertices in \( A \) with \( P \cap Q = \emptyset \). Since both 1 and \( r \) are joined to all of \( A \), \( Q \) together with \( P \) is a cycle \( C_{r+2j+1} \). \( \square \)

**Theorem 7.** Let \( k, m \) be positive integers. There exist \( n_0 = n_0(k, m) \) and \( c = c(k, m) \) such that for every nonbipartite \( G \) on \( n > n_0 \) vertices and minimum degree

\[
\delta > \frac{n}{2(2k+1)} + c,
\]

if \( C_{2s+1} \subset G \), for some \( k \leq s \leq 4k+1 \), then \( C_{2s+2j+1} \subset G \) for every \( j \in [m] \).
Proof. Set
\[ n_0(k, m) = 16(2k + 1)^2 (4k + m + 1) \]
\[ c(k, m) = 2 (4k + m + 1) (8 (2k + 1) (4k + 1) + 1) \]
\[ p = 4 (4k + m + 1) (2k + 1). \]

Let \( C \) be an odd cycle of order \( 2s + 1 \), say \( C = 1, \ldots, 2s + 1, 1 \) and assume the assertion of the theorem does not hold. Clearly, the neighborhood of any vertex cannot contain a subgraph with minimum degree larger than \( 2m + 8k \). Indeed, if \( X \subset N_i \) is a set of vertices and \( H \) is a graph on \( X \) with minimum degree larger than \( 2m + 8k \), then, from Theorem 4, \( H \) contains a path of length at least \( 2m + 8k + 2 \) and consequently \( G \) contains odd cycles of all orders up to \( 2m + 8k + 3 \geq 2m + 2s + 1 \).

Next suppose \( |N_i \cap N_j| \geq p \) for some \( i, j \) with \( 1 \leq i < j \leq 2s + 1 \) and select the path of even length, say \( 2d (1 \leq d \leq s) \), joining \( i \) and \( j \) along \( C \). Apply Lemma 6 with \( c = 1/(4k + 2) \), \( r = 2d \) and \( l = 4k + m \). Since \( 2d \leq 2s \leq 8k + 2 \), we have
\[ p = 4 (4k + m + 1) (2k + 1) = 2 (4k + m) + 8k + 4 > \frac{2 (4k + m)}{1/(4k + 2)} + 2d \]
and
\[ n > \frac{2}{1/(4k + 2)} \left( \frac{2 (4k + m)}{1/(4k + 2)} + 8k + 4 \right) > \frac{2}{1/(4k + 2)} \left( \frac{2 (4k + m)}{1/(4k + 2)} + 2d \right). \]

Hence, \( G \) contains a cycle \( C_{2d+2j+1} \) for every \( j \in [4k + m] \). Since \( 2d \leq 2s \) and
\[ 2s + 2m + 1 \leq 2d + 2(4k + m) + 1, \]
\( G \) contains \( C_{2s+2j+1} \) for every \( j \in [m] \). Thus, if (2) holds for a pair of distinct vertices \( i \) and \( j \) then the proof is complete. Assume hereafter that \( |N_i \cap N_j| < p \) for every \( i, j \) with \( 1 \leq i < j \leq 2s + 1 \) and for every \( i \in [2s + 1] \), let
\[ X_i = N_i - \left( \bigcup_{j \in C, j \neq i} N_j \right). \]

By definition, the sets \( X_i \) are disjoint and we easily obtain
\[ |X_i| \geq \delta - \left| \bigcup_{j \in C, j \neq i} (N_j \cap N_i) \right| > \delta - 2sp > \frac{n}{2(2k + 1)}. \]
Now for every $i \in [2s + 1]$ select a vertex $v_i \in X_i$ such that

$$|N_{v_i} \cap X_i| = \min_{u \in X_i} |N_u \cap X_i|,$$

i.e., $v_i$ is a vertex with a minimum degree in the graph induced by $X_i$. Let $Y_i = N_{v_i} \setminus (X_i \cup \{i\})$. Since $X_i$ is contained in the neighborhood of $i$, we know from above that $|N_{v_i} \cap X_i| \leq 2m + 8k$. Thus,

$$|Y_i| \geq \delta - (8k + 2m + 1) > \frac{n}{2(2k + 1)} + 4(4k + 1)p. \quad (4)$$

Suppose

$$|Y_i \cap Y_j| \geq p \quad (5)$$

holds for some $i, j$ with $1 \leq i < j \leq 2s + 1$ and select the path of even length, say $2d$ ($d \leq s$), joining $i$ and $j$ along $C$. Just as above we can prove that $G$ contains $C_{2s+2j+1}$ for every $j \in [m]$. So, we may suppose that

$$|Y_i \cap Y_j| < p \text{ for every } i, j \text{ with } 1 \leq i < j \leq 2s + 1. \quad (6)$$

Finally, we shall prove that

$$|X_i \cap Y_j| \leq p \text{ for every } i, j \text{ with } 1 \leq i < j \leq 2s + 1. \quad (7)$$

Indeed, if $|X_i \cap Y_j| > p$ for some $i, j$ with $1 \leq i < j \leq 2s + 1$, select the path of odd length, say $2d + 1$ ($d \leq s$), joining $i$ and $j$ along $C$ and append to it the vertex $v_j$ which was used to define $Y_j$. The new path has length $2d + 2$ and its end-vertices have more than $p$ neighbors in common. Just as above we obtain that $G$ contains $C_{2s+2j+1}$ for every $j \in [m]$. Let

$$Z_i = Y_i - \left( \bigcup_{j \in C, j \neq i} Y_j \right) \cup \left( \bigcup_{j \in C} X_i \right)$$

From (6), (7) and (4) we obtain for every $i \in [2s + 1]$

$$|Z_i| \geq |Y_i| - 4sp \geq \frac{n}{2(2k + 1)} + 4(4k + 1)p - 4sp > \frac{n}{2(2k + 1)}. \quad (8)$$

Consider the family of $4s + 2$ sets $X_i$ and $Z_i$ ($i = 1, ..., 2s + 1$). By definition, all these sets are pairwise disjoint, a contradiction in view of (3), (8) and $4s + 2 \geq 4k + 2$. \qed

### 2.1 Structural description of the extremal graphs

The purpose of this section is twofold; on the one hand, we show that Theorem 7 is essentially tight, on the other hand, we give conditions for a graph to be nearly bipartite.

We start by a description of two classes of extremal graphs. Fix two positive integers $k$ and $\delta$ ($\delta \geq 5$),
Example 8. Select $2k + 1$ pairwise vertex disjoint complete bipartite graphs $K_{\delta,\delta}$; choose exactly one vertex from each of them and join the selected $2k + 1$ vertices completely. Denote the resulting graph by $G_{2k+1}$ (Cf. Fig. 1.).

Clearly, $G_{2k+1}$ has $2\delta(2k + 1)$ vertices and $\delta(G_k) = \delta$; it is easy to prove that the order of the largest odd cycle of $G_{2k+1}$ is $2k + 1$.

Example 9. Select $2k + 1$ pairwise vertex disjoint complete bipartite graphs $K_{\delta,\delta}$; choose $3k$ distinct vertices from them so that:

- a) the selected vertices can be partitioned into $k$ triples so that any triple has at most one vertex in common with any component.

- b) if we add the three edges into each triple, the resulting graph is connected.

Denote this graph by $H_{2k+1}$ (Cf. Fig. 2.)
Clearly, $H_{2k+1}$ has $2\delta(2k+1)$ vertices and $\delta(H_{2k+1}) = \delta$; it is easy to prove that the only odd cycles of $H_{2k+1}$ are its $k$ triangles, and the total number of their vertices is obviously $3k$.

**Theorem 10.** Let $k \geq 1$ be a fixed integer and $G$ be a graph of $n \geq 10(2k+1)^2$ vertices with $\delta = \delta(G) \geq \frac{n}{2k+2}$ such that every odd cycle in $G$ has length at most $2k+1$. Then the structure of $G$ is known and there is a set $S$ of at most $2k$ vertices and a set $E_0$ of at most $k^2$ edges in $G$ such that $G - S$ and $G - E_0$ are bipartite graphs. If we have to delete exactly $2k-1$ vertices or exactly $k^2$ edges then $G$ is isomorphic to $G_{2k+1}$, (see Example 8 above.) Furthermore, the union of all odd cycles of $G$ has at most $3k$ vertices, and this is also sharp, in view of $H_{2k+1}$, (see Example 9 above.)

**Remark.** The edge form of Theorem 10 has a separate version for $\delta = \delta(G) \geq \frac{n}{4k}$. Then there is a set $E_0$ of at most $k(k-1)$ edges in $G$ such that $G - E_0$ is a bipartite graph. If we have to delete exactly $k(k-1)$ edges then $G$ must be isomorphic to $G_{2k}$. The proof is very similar to that of Theorem 10, so we omit the details.

**Proof.** Consider the block decomposition of $G$ into maximal 2-connected subgraphs or $K_2$'s such that any two blocks share at most one vertex. We shall consider separately the case when $G$ is a block itself, i.e., $G$ is 2-connected; we shall show that then $G$ is bipartite. We distinguish two types of blocks:

Type 1: Every vertex in the block is a cutvertex.

Type 2: There is a vertex in the block that is not a cutvertex.

**Claim 11.** A block $B$ of type 2 has at least $\delta + 1$ vertices.

**Proof.** This is obvious since any vertex in $B$, which is not a cutvertex, has all its neighbors in $B$.

**Claim 12.** A block $B$ of type 1 has at most $\frac{n}{\delta+1}$ vertices.

**Proof.** Delete the edges of $B$. The resulting graph has $|B|$ components. Every component has a vertex not belonging to $B$ and it has all neighbors in the component, so every component has at least $\delta + 1$ vertices.

In the light of Claims 11 and 12, we call the blocks of Type 1 and 2 small and big, respectively.

Let $x_1, x_2, ..., x_t$ be the cutvertices in $G$ with multiplicities $m_1, ..., m_t$, respectively, where the multiplicity of a vertex is the number of blocks containing it. Further, let $B_1, B_2, ..., B_s$ be the blocks in $G$ and let $l_i$ be the number of cutvertices contained in $B_i$ for $i = 1, ..., s$.

**Claim 13.** Every cutvertex is contained in a big block. Furthermore,

$$\sum_{m_i \geq 2} (m_i - 2) + \sum_{l_i \geq 2} (l_i - 2) < \frac{n}{\delta + 1},$$

in particular $l_i < \frac{n}{\delta + 1} + 2$. 

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Proof. We prove the first statement by contradiction. Suppose that \( x \) is a cutvertex not contained in any big block. Then the union of the blocks containing \( x \) has at least \( \delta + 1 \) vertices. Since these at least \( \delta \) neighbors of \( x \) belong to blocks of Type 1, they all are cutvertices. In the tree structure of the blocks (the vertices of the tree are the blocks and the cutvertices of G, the edges are block-cutvertex incidences), each branch starting from \( x \) and continued via these cutvertices contains an endblock with one cutvertex, i.e., a big block. Therefore, we have at least \( \delta \) big blocks. But we have at most \( \frac{n}{\delta + 1} \) big blocks from Claim 11, so \( \delta \leq \frac{n}{\delta + 1} \), a contradiction.

Furthermore, the tree structure of the block decomposition implies that the number of (big) endblocks is at least

\[
\sum_{m_i>2} (m_i - 2) + \sum_{l_i>2} (l_i - 2) + 2
\]

which implies the second statement, again from Claim 11.

Claim 14. Every big block is bipartite and has at least \( 2\delta \) vertices. In particular, if \( G \) is 2-connected then \( G \) is bipartite.

Proof. Suppose that a big block \( B \) contains an odd cycle \( C \) (of at most \( 2k+1 \) vertices). From Claim 13, \( B \) contains a set \( S \) of at most \( \frac{n}{\delta + 1} + 1 \) cutvertices. Then every vertex in \( B - V(C) - S \) has degree at least \( \delta - (2k + 1) - (\frac{n}{\delta + 1} + 1) \geq 4k - 3 \) in \( G[B - V(C) - S] \), so there is a cycle \( C_0 \) of length \( 4k - 2 \) or more. The block \( B \) is 2-connected, so there are two vertex disjoint paths \( P_1 \) and \( P_2 \) from \( C \) to \( C_0 \), say from \( u_1 \) to \( v_1 \) and from \( u_2 \) to \( v_2 \). Then the two paths \( Q_1, Q_2 \) joining \( u_1 \) and \( u_2 \) in \( C \) have lengths of different parities. Then take the union of a path of at least \( 2k \) vertices joining \( v_1 \) and \( v_2 \) in \( C_0 \), the paths \( P_1, P_2 \) and the path from \( \{Q_1, Q_2\} \) which makes the resulting cycle odd. But the resulting cycle has length more than \( 2k + 1 \), a contradiction.

Now we prove that both color classes \( X \) and \( Y \) of a big block \( B \), have at least \( \delta \) vertices. Suppose that \( |X| < \delta \). Then every vertex \( y \in Y \) has some neighbors not in \( B \), so it is a cutvertex. But, from Claim 11, \( B \) contains a vertex that is not a cutvertex and then it must be in \( X \). But its neighbors are in \( Y \), so \( l_i \geq |Y| \geq \delta \), which contradicts Claim 13.

Remark. Notice that if two big blocks share a cutvertex then we may add edges joining these blocks so that they remain bipartite and get one big block from these two ones. So, we may assume that the big blocks are pairwise vertex-disjoint and the number of them is at most \( \frac{n}{2\delta} \).

Let \( b \) and \( s \) denote the number of big and small blocks, respectively. We have \( b \leq 2k + 1 \) from Claim 14. Let \( T \) be the intersection bipartite graph of big and small blocks: the \( b+s \) vertices are the blocks and a big and a small block is joined by an edge iff they share a cutvertex. Naturally, \( T \) is a tree with \( b+s \) vertices and \( \sum(m_i - 1) \geq t \) edges, so \( b+s \geq t+1 \) or equivalently, \( t - 2s \leq b - s - 1 \leq \frac{n}{2\delta} - 2 \leq 2k - 1 \). If we delete all but two vertices from each small block then we deleted at most \( t - 2s \leq \frac{n}{2\delta} - 2 \leq 2k - 1 \) vertices from \( G \) and made it bipartite. If equality holds then we have one small block of at least
3 vertices and ⌈n/2⌉ big blocks. Similarly, if we delete at most \(\binom{m/2}{2} + \binom{m/2}{2}\) edges from a small block of \(m\) vertices then we can make it bipartite even if it was a clique. Easy calculations show that the total number of edges to be deleted is maximum if we have just one small block which is a clique of \(2k + 1\) vertices and that we have maximum total number of vertices in the small blocks of at least 3 vertices if we have \(k\) triangles.

From Theorems 1, 5, and 6 we obtain the following corollary.

**Corollary 15.** Let \(k\) be a fixed positive integer and let \(G\) be a nonbipartite graph on \(n\) vertices with

\[\delta(G) \geq \frac{2n}{8k+4} + c(k),\]

such that \(G\) contains no \(C_{8k+3}\). If \(n\) is sufficiently large then \(G\) contains no \(C_{2i+1}\) for \(i \geq k\). Further if

\[\delta(G) = \frac{2n}{8k+4},\]

then \(G\) may contain precisely the odd cycles \(C_3, C_5, ..., C_{2k+1}\). □

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**References**


E-mail address: E. Győri, gyori@renyi.hu

E-mail address: V. Nikiforov, vnikifrv@memphis.edu

E-mail address: R. H. Schelp, rschelp@memphis.edu