On the second largest eigenvalue of the signless Laplacian

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Abstract

Let $G$ be a graph of order $n$, and let $q_1(G) \geq \cdots \geq q_n(G)$ be the eigenvalues of the $Q$-matrix of $G$, also known as the signless Laplacian of $G$. We give a necessary and sufficient condition for the equality $q_k(G) = n - 2$, where $1 < k \leq n$. In particular, this result solves an open problem raised by Wang, Belardo, Huang and Borovicanin.

We also show that

$$q_2(G) \geq \delta(G)$$

and determine all graphs for which equality holds.

Keywords: signless Laplacian; second largest eigenvalue; eigenvalue bounds.

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1 Introduction and main results

Given a graph $G$, write $A$ for the adjacency matrix of $G$ and let $D$ be the diagonal matrix of the row-sums of $A$, i.e., the degrees of $G$. The matrix $Q(G) = A + D$, called the signless Laplacian or the $Q$-matrix of $G$, has been intensively studied recently; see, e.g., Cvetković [3] for a comprehensive survey.

As usual, we shall index the eigenvalues of $Q(G)$ in non-increasing order and denote them as $q_1(G), q_2(G), \ldots, q_n(G)$. Also, we shall write $\overline{G}$ for the complement of $G$.

This paper is about the second largest $Q$-eigenvalue of a graph. Some notable contributions to this area have been made by Yan [16], Cvetković, Rowlinson and Simić [4], Wang et al. [15], Das [7], [8], and Aouchiche, Hansen and Lucas [1].

First, in [16], Yan has proved that if $G$ is a graph of order $n \geq 2$, then $q_2(G) \leq n - 2$. It is easy to see that equality holds when $G$ is the complete graph, but there are many other graphs with this property, see, e.g., [1] and [15] for particular examples. Rather naturally, the authors of [15] raised the problem to characterize all graphs $G$ of order $n \geq 2$ such that

$$q_2(G) = n - 2. \quad (1)$$

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The first result of this paper gives a complete solution to this problem, but in fact our methods allow to answer a more general question. To state these results, we need the following definition.

**Definition** A connected bipartite graph is called **balanced** if the sizes of its vertex classes are equal, and **unbalanced** otherwise. An isolated vertex is considered to be an unbalanced bipartite graph with an empty vertex class.

**Theorem 1** If $G$ is a graph of order $n \geq 2$, then $q_2(G) = n - 2$ if and only if $\overline{G}$ has a balanced bipartite component or at least two bipartite components.

Note that if $G$ is a graph of order $n$ and $1 < k \leq n$, then $q_k(G) \leq n - 2$, and since equality is attained for the complete graph it is natural to ask a more general question:

*Given $k \geq 2$, for which graphs $G$ of order $n$ it is true that $q_k(G) = n - 2$?*

In other words, how the structure of a graph of order $n$ relates to the multiplicity of the $Q$-eigenvalue $n - 2$? Our approach allows to specify this relation precisely.

**Theorem 2** Let $1 \leq k < n$ and let $G$ be a graph of order $n$. Then $q_{k+1}(G) = n - 2$ if and only if $\overline{G}$ has either $k$ balanced bipartite components or $k + 1$ bipartite components.

We conclude the paper by comparing $q_2(G)$ to the minimum degree $\delta(G)$. In [7], Das proved that

$$q_2(G) \geq \overline{d}(G) - 1 \text{ and } q_2(G) \geq \Delta_2(G) - 1,$$

where $\overline{d}(G)$ and $\Delta_2(G)$ are the average and the second largest degrees of $G$, respectively. In the light of these inequalities the bound given below looks easy, but its proof is not immediate and it is sharp for many different kinds of graphs. To state the result, write $K_{n_1,n_2,\ldots,n_r}$ for the complete $r$-partite graph with class sizes $n_1, \ldots, n_r$.

**Theorem 3** If $G$ is a noncomplete graph of order $n$, then

$$q_2(G) \geq \delta(G).$$

Equality holds if and only if $G$ is one of the following graphs: a star, a complete regular multipartite graph, the graph $K_{1,3,3}$, or a complete multipartite graph of the type $K_{1,\ldots,1,2,\ldots,2}$.

# 2 Notation and preliminary results

In general, our notation follows [2]; thus, for a graph $G$ we write $V(G)$ for the vertex set of $G$ and $E(G)$ for the edge set of $G$. We write $\overline{G}$ stands for the complement of $G$, and $K_n$ for the complete graph on $n$ vertices.

Furthermore, given a Hermitian matrix $A$ of order $n$, we index its eigenvalues as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. We retain the same notation for the eigenvalues of the self-adjoint operator defined by the matrix $A$. The inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{j}_n$ stands for the $n$-vector of all ones.

Recall that Desao and Rao in [9], Proposition 2, proved the important fact that $0$ is an eigenvalue of $Q(G)$ if and only if $G$ has a bipartite component. In fact, as shown in [5], Corollary 2.2, the following precise statement holds.
Theorem 4 ([5], [9])  Given a graph $G$, the multiplicity of 0 as eigenvalue of $Q(G)$ is equal to the number of bipartite components of $G$.

In our proofs, we shall often use the following basic fact about $Q(G)$: if $G$ is a graph order $n$ and $x = (x_1, \ldots, x_n)$ is an $n$-vector, then

$$
\langle Q(G)x, x \rangle = \sum_{uv \in E(G)} (x_u + x_v)^2. \tag{2}
$$

Also, in the proofs of Theorem 1 and Theorem 2, we shall use Weyl’s inequalities for eigenvalues of Hermitian matrices. Although these fundamental inequalities have been known for almost a century, it seems that their equality case was first established only recently, by So in [13], and his work was inspired by the paper of Ikebe, Inagaki and Miyamoto [12].

For convenience we state below the complete theorem of Weyl and So.

Theorem 5 ([13])  Let $A$ and $B$ be Hermitian matrices of order $n$, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then

$$
\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A + B), \text{ if } i + j \geq n + 1, \tag{3}
$$

$$
\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B), \text{ if } i + j \leq n + 1. \tag{4}
$$

In either of these inequalities equality holds if and only if there exists a nonzero $n$-vector that is an eigenvector to each of the three involved eigenvalues.

Theorem 5 is crucial for our proof of Theorem 1, but it is not general enough for the more complicated Theorem 2. Therefore, we need a strengthening of Theorem 5 in a particular situation. To begin with, note that Theorem 5 can be stated equivalently if we replace “Hermitian matrices” by “self-adjoint linear operators”; indeed, the latter setup seems even more natural.

Proposition 6  Let $2 \leq k < n$ and $A$ and $B$ be self-adjoint operators of order $n$. If for every $s = 2, \ldots, k$,

$$
\lambda_s(A) + \lambda_n(B) = \lambda_s(A + B), \tag{5}
$$

then there exist $k - 1$ nonzero orthogonal $n$-vectors $x^1, \ldots, x^{k-1}$ such that

$$
Ax^{s-1} = \lambda_s(A)x^{s-1}, \quad Bx^{s-1} = \lambda_n(B)x^{s-1}, \quad \text{and} \quad (A + B)x^{s-1} = \lambda_s(A + B)x^{s-1} \tag{6}
$$

for every $s = 2, \ldots, k$.

Proof  Our proof is by induction on $k$. For $k = 2$ the assertion follows from Theorem 5 since we have to find a single vector satisfying the requirements (6). Assume now that $k > 2$ and that the assertion holds for $2 \leq k' < k$. By Theorem 5, there exists a nonzero vector $x = x^{k-1}$ such that

$$
Ax^{k-1} = \lambda_2(A)x^{k-1}, \quad Bx^{k-1} = \lambda_n(B)x^{k-1} \quad \text{and} \quad (A + B)x^{k-1} = \lambda_2(A + B)x^{k-1}.
$$
Write $H$ for the orthogonal complement of $x^{k-1}$. Since $A$ and $B$ are self-adjoint, $H$ is an invariant subspace of the three operators $A$, $B$ and $A + B$. Set $A' = A|H$ and $B' = B|H$; then clearly $A' + B' = (A + B)|H$. Note that

$$
\lambda_1 (A') = \lambda_1 (A), \quad \lambda_2 (A') = \lambda_3 (A), \quad \ldots, \quad \lambda_{n-1} (A') = \lambda_n (A),
$$

$$
\lambda_1 (B') = \lambda_1 (B), \quad \lambda_2 (B') = \lambda_2 (B), \quad \ldots, \quad \lambda_{n-1} (B') = \lambda_{n-1} (B),
$$

$$
\lambda_1 (A' + B') = \lambda_1 (A + B), \quad \lambda_2 (A' + B') = \lambda_3 (A + B), \quad \ldots, \quad \lambda_{n-1} (A' + B') = \lambda_n (A + B).
$$

Hence, equalities (5) imply that

$$
\lambda_s A' + \lambda_{n-1} B' = \lambda_s (A' + B'),
$$

for $s = 2, \ldots, k - 1$. Therefore, there exist $k - 2$ nonzero orthogonal vectors $y^1, \ldots, y^{k-2}$ in $H$ such that

$$
A'y^{s-1} = \lambda_s (A') y^{s-1}, \quad B'y^{s-1} = \lambda_{n-1} (B') y^{s-1}, \quad \text{and} \quad (A' + B') y^{s-1} = \lambda_s (A' + B') y^{s-1}
$$

Considering $H$ as a subspace of $C^2$, the vectors $y^1, \ldots, y^{k-2}$ correspond to $n$-vectors $x^1, \ldots, x^{k-2}$, which together with $x^{k-1}$ have the desired properties. This completes the induction step and the proof of the proposition.

We note that the above proposition is tailored to our needs; clearly other generalizations in the same vein are possible.

3 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 Assume first that $q_2(G) = n - 2$. Applying Weyl’s inequality (3), we find that

$$
q_2 (G) + q_n (\overline{G}) \leq q_2 (K_n).
$$

(7)

Since $q_2 (K_n) = n - 2$, we see that $q_n (\overline{G}) = 0$ and Theorem 4 implies that $\overline{G}$ has a bipartite component. Also, since equality holds in (7), by Theorem 5, there exists a unit vector $x = (x_1, \ldots, x_n)$ that is an eigenvector to each of the eigenvalues $q_2 (G), q_n (\overline{G})$ and $q_2 (K_n)$. The latter implies that $\sum_{i=1}^{n} x_i = 0$ as $x$ is orthogonal to the eigenspace of $q_1 (K_n)$, which is $Span (j_n)$.

Using (2), we see that

$$
0 = q_n (\overline{G}) = \langle Q(\overline{G}) x, x \rangle = \sum_{ij \in E(\overline{G})} (x_i + x_j)^2.
$$

(8)

Therefore, if $\overline{G}$ has just one bipartite component, say $H$, we have $x_w = 0$ for all vertices $w \in V(G) \setminus V(H)$ and $x_u = -x_v$ for each edge $uv \in E(H)$. This means that for every $u \in V(H)$ the entry $x_u$ takes one of two possible values, which have opposite signs. Now the condition $\sum_{i=1}^{n} x_i = 0$ implies that the vertex classes of $H$ are equal in size and so $H$ is balanced. This completes the proof of the “only if” part of the theorem.
Assume now that $\overline{G}$ has a balanced bipartite component, and let $U$ and $W$ be its vertex classes. Define a vector $x = (x_1, \ldots, x_n)$ as

$$x_u = \begin{cases} 1, & \text{if } u \in U; \\
-1, & \text{if } u \in W; \\
0, & \text{if } u \in V(G) \setminus (U \cup W). \end{cases}$$

Since $|U| = |W|$, we see that $q_n(\overline{G}) \|x\|^2 = \langle Q(\overline{G}) x, x \rangle = 0$ and so $x$ is an eigenvector to $q_n(\overline{G})$. Also, $\sum_{i=1}^{n} x_i = 0$ and so $x$ is orthogonal to $\text{Span}(j_n)$; therefore $x$ is an eigenvector to $q_2(K_n)$. Hence $n - 2$ is an eigenvalue to $Q(G)$ with eigenvector $x$. If $G$ is connected, the Perron-Frobenius theorem implies that $x$ is not an eigenvector to $q_1(G)$ because it has negative entries; therefore, $q_2(G) = n - 2$. If $G$ is not connected, then $\overline{G}$ is a connected balanced bipartite graph and so $G = 2K_{n/2}$; therefore $q_2(G) = n - 2$.

Let now $\overline{G}$ have two bipartite components, which can be assumed unbalanced as otherwise the proof is completed by the previous argument. Denote the vertex classes of the one component by $U, W$ and the parts of the other by $X, Y$, and define a vector $x = (x_1, \ldots, x_n)$ as

$$x_u = \begin{cases} 1, & \text{if } u \in U; \\
-1, & \text{if } u \in W; \\
\frac{|W| - |U|}{|X| - |Y|}, & \text{if } u \in X; \\
\frac{|U| - |W|}{|X| - |Y|}, & \text{if } u \in Y; \\
0, & \text{if } u \in V(G) \setminus (U \cup W \cup U \cup W). \end{cases}$$

Since for each edge $uv \in E(\overline{G})$, we have $x_u = -x_v$, it turns out $q_n(\overline{G}) \|x\|^2 = \langle Q(\overline{G}) x, x \rangle = 0$ and $x$ is an eigenvector to $q_n(\overline{G})$. Also, we find that

$$\sum_{i=1}^{n} x_i = |U| - |W| + |X| \frac{|W| - |U|}{|X| - |Y|} + |Y| \frac{|U| - |W|}{|X| - |Y|} = 0,$$

and so $x$ is orthogonal to $\text{Span}(j_n)$ and therefore $x$ is an eigenvector to $q_2(K_n)$. Hence $n - 2$ is an eigenvalue to $Q(G)$ with eigenvector $x$. Since $\overline{G}$ has at least two components, $G$ is connected and the Perron-Frobenius theorem implies that and $x$ is not an eigenvector to $q_1(G)$ because it has negative entries; therefore $q_2(G) = n - 2$. This completes the proof of the theorem.

**Proof of Theorem 2** Note that Theorem 1 covers the case $k = 1$, so we shall assume that $k > 1$. For convenience we split the theorem into two statements:

(A) If $q_{k+1}(G) = n - 2$, then $\overline{G}$ has either $k$ balanced bipartite components or at least $k + 1$ bipartite components;

(B) If $\overline{G}$ has either $k$ balanced bipartite components or $k + 1$ bipartite components, then $q_{k+1}(G) = n - 2$.

First we prove (A). If $q_{k+1}(G) = n - 2$, then obviously

$$q_2(G) = \cdots = q_{k+1}(G) = n - 2,$$

4
and so, for \( i = 2, \ldots, k + 1 \), Weyl’s inequalities (3) and (4) imply that

\[
q_i(G) + q_n(\overline{G}) \leq q_i(K_n) = n - 2.
\]

We see that for every \( i = 2, \ldots, k + 1 \), equality holds throughout (9). In view of this fact, Proposition 6 implies that there exist \( k \) nonzero orthogonal \( n \)-vectors \( \mathbf{x}^1, \ldots, \mathbf{x}^k \) such that for \( s = 1, \ldots, k \),

\[
Q(G)\mathbf{x}^s = q_{s+1}\mathbf{x}^s, \quad Q(\overline{G})\mathbf{x}^s = q_n(\overline{G})\mathbf{x}^s, \quad \text{and} \quad Q(K_n)\mathbf{x}^s = q_{s+1}(K_n)\mathbf{x}^s.
\]

These equalities in particular imply that

\[
q_n(\overline{G}) = \cdots = q_{n+1-k}(\overline{G}) = 0.
\]

Hence, by Theorem 4, \( \overline{G} \) has at least \( k \) bipartite components. For every \( s = 1, \ldots, k \), set \( \mathbf{x}^s = (x^s_1, \ldots, x^s_n) \), and note that \( \sum_{i=1}^{n} x^s_i = 0 \) since \( \mathbf{x}^s \) is orthogonal to the eigenspace of \( q_1(K_n) \), which is \( \text{Span}(\mathbf{j}_n) \).

To complete the proof of (A) we have to show that if \( \overline{G} \) has exactly \( k \) bipartite components, then they are all balanced. For every \( s = 1, \ldots, k \), let \( U_s \) and \( W_s \) be the vertex classes of the \( s \)th bipartite component of \( \overline{G} \) and write \( V_0 \) for the set of vertices of \( \overline{G} \) that do not belong to any bipartite component of \( \overline{G} \). Since

\[
0 = q_{n+1-k}(\overline{G}) = \langle Q(\overline{G})\mathbf{x}^s, \mathbf{x}^s \rangle = \sum_{ij \in E(\overline{G})} (x^s_i + x^s_j)^2
\]

for every edge \( uv \in E(\overline{G}) \), we have \( x^s_u = -x^s_v \). It follows that \( x^s_u = 0 \) if \( u \in V_0 \), and \( x^s_i = x^s_j \) if \( i \) and \( j \) belong to the same \( U_i \) or the same \( W_i \). Hence for every \( s = 1, \ldots, k \), there exist \( k \) numbers \( a^s_1, \ldots, a^s_k \), such that

\[
x^s_u = \begin{cases} 
a^s_i, & \text{if } u \in U_i; 
-a^s_i, & \text{if } u \in W_i; 
0, & \text{if } u \in V_0.
\end{cases}
\]

Now from \( \sum_{i=1}^{n} x^s_i = 0 \) we see that \( p_1a^s_1 + \cdots + p_ka^s_k = 0 \), where \( p_i = |W_i| - |U_i|, 1 \leq i \leq k \).

Let \( B \) be the \( k \times n \) matrix whose rows are the vectors \( \mathbf{x}^1, \ldots, \mathbf{x}^k \). Since \( \text{rank}(B) = k \), there exists \( k \) independent columns of \( B \), say the columns \( \mathbf{c}_1, \ldots, \mathbf{c}_k \) corresponding to the vertices \( u_1, \ldots, u_k \).

Since \( \mathbf{c}_1, \ldots, \mathbf{c}_k \) are nonzero, none the vertices \( u_1, \ldots, u_k \) belongs to \( V_0 \) and so each one of them belongs to a bipartite component. But since \( \mathbf{c}_1, \ldots, \mathbf{c}_k \) are linearly independent, each one of the vertices \( u_1, \ldots, u_k \) belongs to a different component. Define the numbers \( q_1, \ldots, q_k \) by

\[
q_i = \begin{cases} 
p_i, & \text{if } u_i \in \bigcup_{j=1}^{k} U_j; 
-p_i, & \text{if } u_i \in \bigcup_{j=1}^{k} W_j;
\end{cases}
\]

and let

\[
\mathbf{y} = (y_1, \ldots, y_n) = q_1\mathbf{c}_1 + \cdots + q_k\mathbf{c}_k.
\]

The definition (11) implies that for every \( i = 1, \ldots, k \),

\[
q_i\mathbf{c}_i = (p_i a^1_i, p_i a^2_i, \ldots, p_i a^k_i)^T.
\]
Then, for every $s = 1, \ldots, k$,  
\[ y_s = p_1 a_1^s + \cdots + p_k a_k^s = 0, \]
implies that $y = 0$ and so $q_1 = \cdots = q_k = 0$. Thus all bipartite components of $G_s$ are balanced, completing the proof of (A).

Now let us prove (B). Suppose that $G$ has $k$ balanced bipartite components, say with vertex classes $U_i$ and $W_i$, $i = 1, \ldots, k$. For every $s = 1, \ldots, k$, define a vector $x^s = (x_1^s, \ldots, x_n^s)$ by  
\[ x_u^s = \begin{cases} 
1, & \text{if } u \in U_s; \\
-1, & \text{if } u \in W_s; \\
0, & \text{if } u \in V(G) \setminus (U_s \cup W_s). 
\end{cases} \]

Since $|U_s| = |W_s|$, we see that $\langle Q(G) x^s, x^s \rangle = 0$; also, $\sum_{j=1}^n x_j^s = 0$ and so $x^s$ is an eigenvector to $q_{s+1}(K_n)$. Therefore, $n - 2$ is an eigenvalue to $Q(G)$ with eigenvector $x^s$. As the vectors $x^1, \ldots, x^k$ are orthogonal, we see that that $n - 2$ is an eigenvalue of $Q(G)$ with multiplicity at least $k$. To complete the proof in this case, note that none of the vectors $x^1, \ldots, x^k$ can be an eigenvector to $q_1(G)$ since $G$ is connected and each of the vectors $x^1, \ldots, x^k$ has negative entries.

Now assume that $G$ has $k+1$ bipartite components, let $U_s$ and $W_s$ be the vertex classes of the $s$’th bipartite component of $G$ and set $p_s = |W_s| - |U_s|$. Write $V_0$ for the set of all vertices of $G$ that do not belong to any bipartite component of $G$.

To complete the proof of (B), we shall show that there exists $k$ linearly independent vectors $y^1, \ldots, y^k$, each orthogonal to $j_n$ and satisfying $\langle Q(G) y^i, y^i \rangle = 0$. Indeed, in this case each $y^i$ is an eigenvector to $q_2(K_n) = n - 2$ and to $q_n(G) = 0$, implying that $Q(G) y^i = (n - 2) y^i$. Hence $n - 2$ is an eigenvalue of $Q(G)$ with multiplicity at least $k$.

Consider the $k$-dimensional linear space $L$ of all $(k+1)$-vectors $(a_1, \ldots, a_{k+1})$ satisfying  
\[ p_1 a_1 + \cdots + p_{k+1} a_{k+1} = 0 \]
and choose $k$ linearly independent vectors $a_1, \ldots, a^k$ in $L$. Now, for every $s = 1, \ldots, k$, let $a^s = (a_1^s, \ldots, a_{k+1}^s)$ and define the $n$-vector $y^s = (y_1^s, \ldots, y_n^s)$ by  
\[ y_u^s = \begin{cases} 
0, & \text{if } u \in U_s; \\
a_1^s, & \text{if } u \in W_s; \\
-a_i^s, & \text{if } u \in V_0. 
\end{cases} \]

We shall show that the vectors $y^1, \ldots, y^k$ satisfy the requirements. Indeed for every $s = 1, \ldots, k$,  
\[ \sum_{u \in V} y_u^s = p_1 a_1 + \cdots + p_{k+1} a_{k+1} = 0; \]
hence, each $y^s$ is orthogonal to $j_n$. Also $y_u^s = -y_u^s$ for every edge $uv \in E(G)$; hence $\langle Q(G) y^s, y^s \rangle = 0$ for every $s = 1, \ldots, k$.

Finally assume that  
\[ c_1 y^1 + \cdots + c_k y^k = 0. \]
For every $i = 1, \ldots, k + 1$, choose a vertex $u \in U_i$ and note that
\[ c_1 a_i^1 + \cdots + c_k a_i^k = c_1 y_u^1 + \cdots + c_k y_u^k = 0, \]
This implies that
\[ c_1 a^1 + \cdots + c_k a^k = 0, \]
and since $a^1, \ldots, a^k$ are linearly independent, it turns out that $c_1 = \cdots = c_k = 0$; hence $y^1, \ldots, y^k$ are also linearly independent. This completes the proof of (B) and of Theorem 2.

**Proof of Theorem 3** Applying Weyl’s inequality (3) we find that
\[ q_2(G) \geq \lambda_2(G) + \delta(G), \]
where $\lambda_2(G)$ is the second largest eigenvalue of the adjacency matrix of $G$. Since $\lambda_2(G) \geq 0$ with equality only for the complete multipartite graphs with possibly isolated vertices (this was proved by Smith in [14]), it follows that $q_2(G) \geq \delta(G)$, and equality is possible only if $G$ is a complete multipartite graph. Let thus $q_2(G) = \delta(G)$ and $G$ be a complete $r$-partite graph with part sizes $n_1 \leq n_2 \leq \cdots \leq n_r$. In this case, $q_2(G) = \delta(G) = n - n_r$. If $r = 2$, it is known, that
\[ q_2(K_{n_1,n_2}) = \begin{cases} 
1, & \text{if } n_1 = 1, \\
n_2, & \text{if } n_1 \geq 2.
\end{cases} \]
Hence $n_1 = 1$ or $n_1 = n_2$, which completes the proof for $r = 2$. Let now $r \geq 3$. If $n_1 \geq 2$, then $G$ contains $K_{n_1,n-n_1}$; and so
\[ n - n_r = q_2(G) \geq n - n_1, \]
implying that $G$ is regular complete multipartite graph.

Let now $n_1 = 1$. If $n_2 \leq 2$, then $G$ contains $K_{2,n-2}$ and so
\[ n - n_r = q_2(G) = n - 2. \]
Therefore $G = K_{1,\ldots,1,\ldots,2}$.

If $n_2 > 2$, then $G$ contains $K_{n_2,n-n_2}$ and so
\[ n - n_r = q_2(G) \geq n - n_2, \]
implying that $G = K_{1,\ldots,1,t,\ldots,t}$ for some $t \geq 2$. We shall show that this is only possible if $t = 2$, or $r = 3$ and $t = 3$.

As proved in [10], the characteristic polynomial of the $Q$-matrix of the $(r + 1)$-partite graph $G = K_{1,\ldots,1,t,\ldots,t}$ satisfies
\[ P_Q(G, x) = (x - tr + t - 1)^{r(t-1)}(x - tr + 2t - 1)^{(r-1)}(x^2 - (2tr - 2t + tr + 1)x + 2t^2r(r - 1)), \]
with roots
\[ tr - t + 1, \quad \frac{3tr - 2t + 1 \pm \sqrt{t^2(r - 2)^2 + 2t(3r - 2) + 1}}{2} \quad \text{and} \quad tr - 2t + 1. \]
Since
\[
\frac{3tr - 2t + 1 + \sqrt{t^2(r - 2)^2 + 2t(3r - 2) + 1}}{2} > tr - t + 1 > tr - 2t + 1,
\]
we see that
\[
q_1(G) = \frac{3tr - 2t + 1 + \sqrt{t^2(r - 2)^2 + 2t(3r - 2) + 1}}{2}.
\]

Also, if \( t > 2 + \frac{1}{r-1} \), one can show that
\[
tr - t + 1 < \frac{3tr - 2t + 1 - \sqrt{t^2(r - 2)^2 + 2t(3r - 2) + 1}}{2},
\]
and so,
\[
q_2(G) = \frac{3tr - 2t + 1 - \sqrt{t^2(r - 2)^2 + 2t(3r - 2) + 1}}{2} > \delta(G).
\]

Finally, if \( t \leq 2 + \frac{1}{r-1} \), we find that \( q_2(G) = tr - t + 1 = \delta(G) \); in that case, when \( r = 2 \), the only feasible graphs are \( K_{1,2,2} \) and \( K_{1,3,3} \); and when \( r \geq 3 \), the feasible graphs are of the type \( K_{1,2,\ldots,2} \), completing the proof.

\( \square \)

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References


