Eigenvalues and extremal degrees in graphs

Vladimir Nikiforov
Department of Mathematical Sciences, University of Memphis,
Memphis TN 38152, USA, email: vnkifrv@memphis.edu

July 2, 2007

Abstract

Let $G$ be a graph with $n$ vertices, $\mu_1 (G) \geq \cdots \geq \mu_n (G)$ be the eigenvalues of its adjacency matrix, and $0 = \lambda_1 (G) \leq \cdots \leq \lambda_n (G)$ be the eigenvalues of its Laplacian. We show that

$$\delta (G) \leq \mu_k (G) + \lambda_k (G) \leq \Delta (G) \quad \text{for all } 1 \leq k \leq n,$$

and

$$\mu_k (G) + \mu_{n-k+2} (\overline{G}) \geq \delta (G) - \Delta (G) - 1 \quad \text{for all } 2 \leq k \leq n.$$

Let $G$ be an infinite family of graphs. We prove that $G$ is quasi-random if and only if $\mu_n (G) + \mu_n (\overline{G}) = o (n)$ for every $G \in G$ of order $n$. This also implies that if $\lambda_n (G) + \lambda_n (\overline{G}) = n + o (n)$ for every $G \in G$ of order $n$, then $G$ is quasi-random.

AMS classification: 15A42, 05C50

Keywords: graph eigenvalues, Laplacian eigenvalues, minimum degree, maximum degree, quasi-random graphs, conditions for quasi-randomness

1 Introduction

Our notation is standard (e.g., see [1], [2], and [5]); in particular, all graphs are defined on the vertex set $\{1, 2, \ldots, n\}$, $G(n)$ stands for a graph of order $n$, and $\overline{G}$ denotes the complement of $G$. Writing $A(G)$ for the adjacency matrix of $G$ and $D(G)$ for the diagonal matrix of its degree sequence, the Laplacian of $G$ is defined as $L(G) = D(G) - A(G)$. If $G = G(n)$, we order the eigenvalues of $A(G)$ as $\mu_1 (G) \geq \cdots \geq \mu_n (G)$ and the eigenvalues of $L(G)$ as $0 = \lambda_1 (G) \leq \cdots \leq \lambda_n (G)$.

In this note we prove that if $G = G(n)$ is a graph with minimum degree $\delta (G)$ and maximum degree $\Delta (G)$, then

$$\delta (G) \leq \mu_k (G) + \lambda_k (G) \leq \Delta (G) \quad \text{for all } 1 \leq k \leq n. \quad (1)$$

This, in turn, implies that

$$\mu_k (G) + \mu_{n-k+2} (\overline{G}) \geq \delta (G) - \Delta (G) - 1 \quad \text{for all } 2 \leq k \leq n, \quad (2)$$
complementing the well-known inequality $\mu_k (G) + \mu_{n-k+2} (G) \leq -1$.

In the second part of this note we give new spectral conditions for quasi-randomness of graphs. Throughout this note we denote by $\mathcal{G}$ an infinite family of graphs. Following Chung, Graham, and Wilson [3], we call a family $\mathcal{G}$ quasi-random, if for every $G \in \mathcal{G}$ of order $n$,

$$\mu_1 (G) = 2e (G) / n + o (n), \quad \mu_2 (G) = o (n), \quad \text{and} \quad \mu_n (G) = o (n).$$

Applying results of [6], we first prove the following theorem.

**Theorem 1** A family $\mathcal{G}$ is quasi-random if and only if

$$\mu_n (G) + \mu_n (\overline{G}) = o (n) \quad (3)$$

for every graph $G \in \mathcal{G}$ of order $n$.

This, in turn, implies the following sufficient conditions for quasi-randomness in terms of Laplacian eigenvalues.

**Theorem 2** If $\mathcal{G}$ is a family such that

$$\lambda_n (G) + \lambda_n (\overline{G}) = n + o (n) \quad (4)$$

for every $G \in \mathcal{G}$ of order $n$, then $\mathcal{G}$ is quasi-random.

Since $\lambda_2 (G) + \lambda_n (\overline{G}) = n$ for every $G = G (n)$, we also obtain the following theorem.

**Theorem 3** If $\mathcal{G}$ is a family such that

$$\lambda_2 (G) + \lambda_2 (\overline{G}) = n + o (n)$$

for every $G \in \mathcal{G}$ of order $n$, then $\mathcal{G}$ is quasi-random.

We leave the extension of the above results to normalized Laplacians to the interested reader.

2 Proofs

**Proof of inequality (1)** Let $u_1, \ldots, u_n$ be orthogonal unit eigenvectors to $\lambda_1, \ldots, \lambda_n$. For every $k = 2, \ldots, n$, the variational characterization of eigenvalues of Hermitian matrices ([5], p. 178-179) implies that

$$\lambda_k (G) = \min_{\|x\|=1, x \perp \text{Span}\{u_1, \ldots, u_{k-1}\}} \langle Lx, x \rangle \quad (5)$$

$$\mu_k (G) = \min_{M \subseteq \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|x\|=1, x \perp M} \langle Ax, x \rangle \right\} \quad (6)$$

2
Let $y$ be such that $\langle Ay, y \rangle$ is maximal subject to $\|y\| = 1$ and $y \perp \text{Span} \{u_1, \ldots, u_{k-1}\}$. Letting $y = (y_1, \ldots, y_n)$, we find that

$$
\lambda_k(G) \leq \langle Ly, y \rangle = \sum_{u \in V(G)} d(u) y_u^2 - \langle Ay, y \rangle \leq \Delta(G) - \max_{\|x\| = 1, x \perp \text{Span} \{u_1, \ldots, u_{k-1}\}} \langle Ax, x \rangle
$$

$$
\leq \Delta(G) - \min_{M \subset \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|x\| = 1, x \perp M} \langle Ax, x \rangle \right\} = \Delta(G) - \mu_k(G),
$$

proving the second inequality of (1). The first inequality is deduced likewise using the dual version of (5) and (6).

\[ \square \]

**Proof of inequality (2)** It is known that $\lambda_k(G) + \lambda_{n-k+2}(\overline{G}) = n$ for all $2 \leq k \leq n$. This, in view of (1), implies that

$$
n + \mu_k(G) + \mu_{n-k+2}(\overline{G}) = \lambda_k(G) + \lambda_{n-k+2}(\overline{G}) + \mu_k(G) + \mu_{n-k+2}(\overline{G})
$$

$$
\geq \delta(G) + \delta(\overline{G}) \geq \delta(G) + n - 1 - \Delta(G),
$$

completing the proof of (2).

\[ \square \]

**Proof of Theorem 1** The necessity of condition (3) is a routine fact, so we shall prove only its sufficiency. Let $G = G(n), e(G) = m$, and set $s(G) = \sum_{u \in V(G)} |d(u) - 2m/n|$. The following results were obtained in [6]

$$
\frac{s^2(G)}{2m^2} \leq \mu_1(G) - 2m/n \leq \sqrt{s(G)},
$$

$$
\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq -1 - 2\sqrt{2s(G)} \quad \text{for all} \quad 2 \leq k \leq n,
$$

$$
\mu_n(G) + \mu_n(\overline{G}) \leq -1 - s^2(G) / (2m^3).
$$

Hence, if (3) holds, (9) implies $\mu_n(G) = o(n), \mu_n(\overline{G}) = o(n)$, and $s(G) = o(n^2)$. Thus, from (7) we obtain $\mu_1(G) = 2m/n + o(n)$. Since $\mu_2(G) + \mu_n(\overline{G}) \leq -1$, inequality (8) implies that $\mu_2(G) = o(n)$, completing the proof.

\[ \square \]

**Proof of Theorem 2** According to Grone and Merris [4], $\lambda_n(G) \geq \Delta(G)$. Thus, (4) implies

$$
n - 1 + \Delta(G) - \delta(G) = \Delta(G) + \Delta(\overline{G}) \leq \lambda_n(G) + \lambda_n(\overline{G}) = n + o(n).
$$

Hence,

$$
\Delta(G) - \delta(G) = \Delta(\overline{G}) - \delta(\overline{G}) = o(n)
$$

and (1) implies

$$
\mu_n(G) = -\lambda_n(G) + \Delta(G) + o(n)
$$

$$
\mu_n(\overline{G}) = -\lambda_n(\overline{G}) + \delta(\overline{G}) + o(n).
$$

Adding these two inequalities, in view of (4), we obtain $\mu_n(G) + \mu_n(\overline{G}) = o(n)$; the assertion follows from Theorem 1.

\[ \square \]

**Acknowledgment** The author is indebted to Béla Bollobás for his kind support.
References


