An extension of Maclaurin’s inequality

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Abstract

Let \( G \) be a graph of order \( n \) and clique number \( \omega \). For every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( 1 \leq s \leq \omega \), set

\[ f_s(G, x) = \sum \{x_i_1 \ldots x_i_s : \{i_1, \ldots, i_s\} \text{ is an } s\text{-clique of } G\}, \]

and let \( \rho_s(G, x) = f_s(G, x) \binom{\omega}{s}^{-1} \). We show that if \( x \geq 0 \), then

\[ \rho_1(G, x) \geq \rho_2^{1/2}(G, x) \geq \cdots \geq \rho_\omega^{1/\omega}(G, x). \]

This extends the inequality of Maclaurin \((G = K_n)\) and generalizes the inequality of Motzkin and Straus. In addition, if \( x > 0 \), for every \( 1 \leq s < \omega \) we determine when \( \rho_s^{1/s}(G, x) = \rho_{s+1}^{1/(s+1)}(G, x) \).

Letting \( k_s(G) \) be the number of \( s \)-cliques of \( G \), we show that the above inequality is equivalent to the combinatorial inequality

\[ \frac{k_1(G)}{\binom{\omega}{1}} \geq \left( \frac{k_2(G)}{\binom{\omega}{2}} \right)^{1/2} \geq \cdots \geq \left( \frac{k_\omega(G)}{\binom{\omega}{\omega}} \right)^{1/\omega}. \]

These results complete and extend earlier results of Motzkin and Straus, Khadzhi-ivanov, Fisher and Ryan, and Petingi and Rodriguez.

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Our graph-theoretic notation follows [1]; in particular, all graphs are defined on the vertex set \( \{1, 2, \ldots, n\} = [n] \) and \( G(n) \) stands for a graph with \( n \) vertices. We write \( \omega(G) \) for the size of the maximal clique of \( G \), \( K_s(G) \) for the set of \( s \)-cliques of \( G \), and \( k_s(G) \) for \( |K_s(G)| \).

For any graph \( G = G(n) \), vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and \( 1 \leq s \leq \omega = \omega(G) \), set

\[ f_s(G, x) = \sum \{x_{i_1} \ldots x_{i_s} : \{i_1, \ldots, i_s\} \in K_s(G)\}. \]
and let \( \rho_s (G, x) = f_s (G, x) \left( \frac{\omega}{s} \right)^{-1} \). The inequality of Maclaurin (see, e.g., [4], p. 52) reads as: if \( G = K_n \) and \( x \geq 0 \), then

\[
\rho_1 (G, x) \geq \rho_2^{1/2} (G, x) \geq \cdots \geq \rho_{\omega}^{1/\omega} (G, x) .
\] (1)

As it turns out, this inequality is valid for any graph \( G \) and any \( x \geq 0 \). Moreover, letting \( x \) to be the vector of all ones, we obtain

\[
\frac{k_1 (G)}{(\frac{\omega}{s})} \geq \left( \frac{k_2 (G)}{(\frac{\omega}{s})} \right)^{1/2} \geq \cdots \geq \left( \frac{k_{\omega} (G)}{(\frac{\omega}{s})} \right)^{1/\omega}
\] (2)

and this, in particular, essentially implies Turán’s [8] and Zykov’s [9] theorems.


In a different approach Eckhoff [2] found exactly \( \max k_s (G) \) for given \( k_2 (G) \) and \( \omega (G) \); up to low-order terms, his result is implied by (2).

Note that the inequality \( \rho_1 (G, x) \geq \rho_2^{1/2} (G, x) \) was proved for any graph \( G \) by Motzkin and Straus [6], so (1) extends their result as well.

It should be noted, however, that the argument of Khadzhiivanov contains a gap and his statement of the cases of equality in (1) is incorrect. Below we give a complete analytical proof of (1) and determine the cases of equality.

Somewhat surprisingly, inequality (2) implies in turn (1), in particular, Turán’s theorem implies Motzkin-Straus’s result. Indeed, since \( f_s (G, x) \) is continuous in \( x \), it suffices to deduce (1) for all \( x \) with positive rational coordinates. Thus, in view of

\[
\rho_s^{1/s} (G, \alpha x) = \alpha \rho_s^{1/s} (G, x) \quad \text{for all } \alpha \geq 0, x \geq 0 ,
\] (3)

it suffices to deduce (1) for all \( x \) with positive integer coordinates. Let \( x_1, \ldots, x_n \) be positive integers and define a graph \( G_x \) as follows: select \( n \) disjoint sets \( V_1, \ldots, V_n \) with \( |V_1| = x_1, \ldots, |V_n| = x_n \), and set \( V (G_x) = \bigcup_{i=1}^{n} V_i \); join \( u \in V_i \) to \( v \in V_j \) if and only if \( ij \in E (G) \). We immediately see that \( \omega (G_x) = \omega (G) \) and \( f_s (G, x) = k_s (G_x) \). Hence, applying (2) to the graph \( G_x \), we see that (1) holds for \( G \) and \( x = (x_1, \ldots, x_n) \), proving the claim.

Thus, (1) is an analytical result with a combinatorial proof and (2) is a combinatorial result with an analytical proof.

**Proof of inequality (1)**

In view of (3), to prove (1) for any \( G = G (n) \) and every \( s \in [\omega - 1] \), it suffices to find \( \max f_{s+1} (G, x) \), subject to \( f_s (G, x) = 1 \). Let

\[
S_s (G) = \{ x : x \in \mathbb{R}^n, x \geq 0 \text{ and } f_s (G, x) = 1 \}
\]

and note that the set \( S_s (G) \) is closed; for \( s \geq 2 \) it is unbounded and therefore, non-compact. Our proof is based on two lemmas. Khadzhiivanov in [5] has overlooked the need of the first one of them.
Lemma 1 For every $G = G(n)$ and $1 \leq s < \omega(G)$, the function $f_{s+1}(G, x)$ attains a maximum on $S_s(G)$.

Proof The lemma is obvious for $s = 1$, since $S_1(G)$ is compact, so we shall assume $s \geq 2$. Our proof is by induction on $n$. Let $n = s + 1$, i.e., $G = K_{s+1}$. For every $x \in S_s(G)$, the AM-GM inequality implies that

$$f_{s+1}(G, x) = x_1 x_2 \ldots x_{s+1} \leq \left( \frac{x_1 \ldots x_s + \ldots + x_2 x_3 \ldots x_{s+1}}{s+1} \right)^{(s+1)/s} = (s+1)^{(s+1)/s}$$

On the other hand, letting $y = (s+1)^{-1/s} (1, \ldots, 1) \in \mathbb{R}^{s+1}$, we see that $f_s(G, y) = 1$ and $f_{s+1}(G, y) = (s+1)^{-(s+1)/s}$. Hence, the assertion holds for $n = s + 1$; assume the assertion holds for any graph with fewer than $n$ vertices.

Suppose first that $G$ has a vertex $v$ that is not contained in any $s$-clique of $G$; let say $v = 1$. We clearly have $f_s(G - v, (x_2, \ldots, x_{s+1})) = f_s(G, x) = 1$ and $f_{s+1}(G, x) = f_{s+1}(G - v, (x_2, \ldots, x_{s+1}))$. Since, by the induction hypothesis, the assertion holds for the graph $G - v$, it holds for $G$ as well. So we may and shall assume that each vertex of $G$ is contained in an $s$-clique.

For all $x \in S_s(G)$ and all $\{i_1, \ldots, i_s\} \in K_s(G)$, we have $x_{i_1} \ldots x_{i_s} \leq f_s(G, x) = 1$. Thus, $x_{i_1} \ldots x_{i_{s+1}} \leq 1$ for every $(s + 1)$-clique $\{i_1, \ldots, i_{s+1}\}$, and consequently, $f_{s+1}(G, x) \leq \binom{n}{s+1}$. Set

$$M = \sup_{x \in S_s(G)} f_{s+1}(G, x)$$

and, for every $i \geq 1$, select $x^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)}) \in S_s(G)$ so that $\lim_{i \to \infty} f_{s+1}(G, x^{(i)}) = M$.

To finish the proof, we shall find $y \in S_s(G)$ with $f_{s+1}(y) = M$. If, for every $t \in [n]$, the sequence $\{x^{(i)}_t\}_{i=1}^\infty$ is bounded, then $\{x^{(i)}\}_{i=1}^\infty$ has an accumulation point $x_0 \in S_s(G)$, and so $f_{s+1}(G, x_0) = M$, completing the proof. Assume now that $\{x^{(i)}_t\}_{i=1}^\infty$ is unbounded for some $t \in [n]$. By assumption, $t \in R$ for some $R \in K_s(G)$; let say $R = \{1, \ldots, s - 1, t\}$. Assume that there exists $c > 0$ such that $x^{(i)}_v > c$ for all $v \in [s - 1]$, $i \geq 1$. Hence, for all $i \geq 1$,

$$M \geq f_s(G, x^{(i)}) \geq x_1^{(i)} \ldots x_{s-1}^{(i)} x_t^{(i)} > c^{s-1} x_t^{(i)}$$

a contradiction, since $\{x_t^{(i)}\}_{i=1}^\infty$ is unbounded. Therefore, for some $v \in [s - 1]$, the sequence $\{x_v^{(i)}\}_{i=1}^\infty$ contains arbitrarily small terms; let say $v = 1$. Note that, for all $i \geq 1$,

$$f_{s+1}(G, x^{(i)}) \leq x_1^{(i)} f_s(G, x^{(i)}) + f_{s+1} \left( G - v, \left( x_2^{(i)}, \ldots, x_n^{(i)} \right) \right)$$

and

$$f_s \left( G - v, \left( x_2^{(i)}, \ldots, x_n^{(i)} \right) \right) \leq f_s \left( G, \left( x_1^{(i)}, \ldots, x_n^{(i)} \right) \right) = 1$$

and

$$f_{s+1}(G, x^{(i)}) = f_s(G, x^{(i)})$$

Now, for each $i \geq 1$, let $x^{(i)} = x_1^{(i)} x_2^{(i)} \ldots x_{s+1}^{(i)}$. Then $x^{(i)} \in S_s(G)$ and $f_{s+1}(G, x^{(i)}) = 1$. Hence, $M = f_{s+1}(G, x^{(i)}) = 1$. This completes the proof.
By the induction hypothesis, the function \( f_{s+1}(G - v, x) \) attains its maximum on \( S_s(G - v) \), let say at \( y = (y_1, \ldots, y_{n-1}) \in S_s(G - v) \), and so
\[
f_{s+1}\left(G - v, \left(x_2^{(i)}, \ldots, x_n^{(i)}\right)\right) \leq f_{s+1}(G - v, y).\]

Hence, in view of (4), we have
\[
f_{s+1}(G - v, y) \leq M \leq x_1^{(i)} + f_{s+1}(G - v, y).
\]

Since \( x_1^{(i)} \) can be arbitrarily small, it follows that \( f_{s+1}(G - v, y) = M \), and so
\[
f_{s+1}(G, (0, y_1, \ldots, y_{n-1})) = f_{s+1}(G - v, y) = M,
\]
completing the proof. \( \square \)

In the proof of the following crucial lemma we apply Lagrange multipliers as suggested by Khadzhiivanov [5].

**Lemma 2** Assume that \( G = G(n) \) is a noncomplete graph, \( 1 \leq s < \omega(G) \), every vertex of \( G \) is contained in some \( s \)-clique, and \( f_{s+1}(G, x) \) attains a maximum, subject to \( x \in S_s(G) \) at some \( y > 0 \). If \( u, v \) are nonadjacent vertices of \( G \), then there exists \( z = (z_1, \ldots, z_n) \in S_s(G) \) such that \( f_{s+1}(G, z) = f_{s+1}(G, y) \) and \( z_u = 0 \).

**Proof** Without loss of generality we shall assume that \( u = 1, v = 2 \). For every \( 1 \leq k \leq \omega(G) \), \( \xi, \eta, \) and \( x = (x_1, \ldots, x_n) \), we have the equality (Taylor’s expansion)
\[
f_k(G, (x_1 + \xi, x_2 + \eta, \ldots, x_n)) = \xi \frac{\partial f_k(G, x)}{\partial x_1} + \eta \frac{\partial f_k(G, x)}{\partial x_2} + f_k(G, x). \tag{5}
\]

Since \( f_{s+1}(G, x) \) attains a maximum at \( y \), subject to \( f_s(G, x) = 1 \), by Lagrange’s method, there exists \( \lambda \) such that \( \partial f_{s+1}(G, y)/\partial x_i = \lambda \partial f_s(G, y)/\partial x_i \) for all \( i \in [n] \). Setting
\[
\xi = -y_1, \ \eta = y_1 \frac{\partial f_s(G, y)}{\partial x_1} / \frac{\partial f_s(G, y)}{\partial x_2},
\]
we see that
\[
\xi \frac{\partial f_s(G, y)}{\partial x_1} + \eta \frac{\partial f_s(G, y)}{\partial x_2} = 0
\]
and
\[
\xi \frac{\partial f_{s+1}(G, y)}{\partial x_1} + \eta \frac{\partial f_{s+1}(G, y)}{\partial x_2} = \lambda \left( \xi \frac{\partial f_s(G, y)}{\partial x_1} + \eta \frac{\partial f_s(G, y)}{\partial x_2} \right) = 0. \tag{6}
\]

Hence, equality (5) with \( k = s \), implies that
\[
f_s(G, (0, y_2 + \eta, y_3, \ldots, y_n)) = f_s(G, (y_1 + \xi, y_2 + \eta, y_3, \ldots, y_n)) = \xi \frac{\partial f_s(G, y)}{\partial x_1} + \eta \frac{\partial f_s(G, y)}{\partial x_2} + f_s(G, y) = 1,
\]

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and so, \( z = (0, y_2 + \eta, y_3, \ldots, y_n) \in S_s(G) \). On the other hand, equality (5) with \( k = s + 1 \) and (6) imply that

\[
    f_{s+1}(G, z) = \xi \frac{\partial f_{s+1}(G, y)}{\partial x_1} + \eta \frac{\partial f_{s+1}(G, y)}{\partial x_2} + f_{s+1}(G, y) = f_{s+1}(G, y),
\]

completing the proof. \( \square \)

To prove (1) we first find \( y \in S_s(G) \) such that \( f_{s+1}(G, y) \geq f_{s+1}(G, x) \) for all \( x \in S_s(G) \). Set

\[
    R = \{ v : v \in V(G), \ y_v > 0 \text{ and } v \text{ is contained in an } s\text{-clique} \};
\]

without loss of generality we may assume that \( G = G[R] \). Applying Lemma 2 iteratively (i.e., using induction on \( n \)), we see that there exists \( y \in S_s(G) \) such that \( f_{s+1}(G, y) \geq f_{s+1}(G, x) \) for all \( x \in S_s(G) \) and the set

\[
    R = \{ v : v \in V(G), \ y_v > 0 \}
\]

induces a complete graph in \( G \); let \( r = |R| \leq \omega \). Maclaurin’s inequality implies that

\[
    (f_{s+1}(K_r, y))^{1/(s+1)} \leq \left( \frac{r}{s+1} \right)^{1/(s+1)} \left( \frac{r}{s} \right)^{1/s} \leq \left( \frac{\omega}{s+1} \right)^{1/(s+1)} \left( \frac{\omega}{s} \right)^{1/s},
\]

and so \( \rho_{s+1}^{1/(s+1)}(G, x) \leq \rho_s^{1/s}(G, x) \) for every \( 1 \leq s < \omega(G) \) and \( x \geq 0 \), completing the proof of (1). \( \square \)

**Cases of equality in (1)**

Let \( G = G(n) \) be a graph, \( 1 \leq s < \omega = \omega(G) \), \( x = (x_1, \ldots, x_n) > 0 \), and \( \rho_s(G, x) \) be defined as above; set

\[
    R_s = \{ v : v \in V(G), \ v \text{ is contained in some } s\text{-clique} \}.
\]

**Theorem 3** The equality \( \rho_{s+1}^{1/(s+1)}(G, x) = \rho_s^{1/s}(G, x) \) holds if and only if \( R_s \) induces a complete \( \omega \)-partite graph and if \( V_1, \ldots, V_\omega \) are the vertex classes of \( G[R_s] \), then \( \sum_{v \in V_i} x_v = \sum_{v \in V_j} x_v \) for all \( 1 \leq i < j \leq \omega \).

**Proof** Assume \( \rho_{s+1}^{1/(s+1)}(G, x) = \rho_s^{1/s}(G, x) \), set \( \overline{R} = G[R_s] \) - the complement of the graph induced by \( R_s \); let \( G_1, \ldots, G_r \) be the components of \( \overline{R} \). Clearly, \( r \leq \omega \); first we shall prove that \( r = \omega \). Assume for simplicity that \( f_s(G, x) = 1 \); hence \( f_{s+1}(G, x) = \left( \frac{\omega}{s+1} \right) \left( \frac{\omega}{s} \right)^{-(s+1)/s} \) and \( x \in S_s(G) \). Applying Lemma 2, preserving the value of \( f_{s+1}(G, x) \), find a vector \( y \in S_s(G) \) with zero coordinates for all but one vertex from each component. Then, by Maclaurin’s inequality, we see that

\[
    \left( \frac{\omega}{s+1} \right) \left( \frac{\omega}{s} \right)^{-(s+1)/s} = f_{s+1}(G, y) \leq \left( \frac{r}{s+1} \right) \left( \frac{r}{s} \right)^{-(s+1)/s},
\]

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and so \( r = \omega \). Clearly \( G_1, \ldots, G_r \) are complete subgraphs of \( \overline{R} \), since otherwise \( K_{\omega+1} \subset G \); hence, \( G[R_s] \) is a complete \( \omega \)-partite graph. Setting \( z_i = \sum_{v \in V_i} x_v \) for every \( i \in [\omega] \), we see that

\[
\left( \frac{\omega}{s+1} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_{s+1} \leq \omega} z_{i_1} \cdots z_{i_{s+1}} \right)^{1/(s+1)} = \rho_{s+1}^{1/(s+1)}(G, x) = \rho_s^{1/s}(G, x)
\]

\[
= \left( \frac{\omega}{s} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_s \leq \omega} z_{i_1} \cdots z_{i_s} \right)^{1/s}.
\]

It is known (see, e.g., [4], p. 52) that equality holds in

\[
\rho_{s+1}^{1/(s+1)}(K_\omega, (z_1, \ldots, z_\omega)) = \rho_s^{1/s}(K_\omega, (z_1, \ldots, z_\omega))
\]

if and only if \( z_1 = \cdots = z_\omega \). Hence, the necessity of the condition is proved. The sufficiency is immediate. □

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**References**


