Regular, pseudo-regular, and almost regular matrices

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Abstract

We give lower bounds on the largest singular value of arbitrary matrices, some of which are asymptotically tight for almost all matrices. To study when these bounds are exact, we introduce several combinatorial concepts. In particular, we introduce regular, pseudo-regular, and almost regular matrices. Nonnegative, symmetric, almost regular matrices were studied earlier by Hoffman, Wolfe, and Hofmeister.

Keywords: largest singular value; nonnegative matrices, regular matrices; pseudo-regular matrices; almost regular matrices

1 Introduction

Let $\Sigma (A)$ be the sum of the entries of a matrix $A$. Hoffman, Wolfe and Hofmeister [3] showed that if $A = (a_{ij})$ is a nonnegative symmetric matrix with positive rowsums $d_1, \ldots , d_n$ and $\mu (A)$ is its largest eigenvalue, then

$$\mu (A) \geq \frac{1}{\Sigma (A)} \sum_{i,j} a_{ij} \sqrt{d_i d_j}$$  (1)

with equality holding if and only if $d_i d_j = \mu^2 (A)$ whenever $a_{ij} > 0$.

The aim of this note is to extend this result in several directions. First, instead of $\mu (A)$, we consider the largest singular value $\sigma (A)$, thereby dropping the requirement that $A$ is symmetric, square, and nonnegative. Second, we present wider classes of lower bounds on $\sigma (A)$ some of which are asymptotically tight for almost all matrices. Finally, we study when these lower bounds are exact, thus introducing regular, pseudo-regular, and almost regular matrices. We also introduce a few combinatorial concepts to support the study of structural properties of arbitrary matrices.

For basic notation and definitions see [2]. Specifically, we call a matrix scalar if it is a scalar multiple of a nonnegative matrix. Also, we write $j_m$ for the vector of $m$ ones.
2 Main results

Let $A = (a_{ij})$ be an $m \times n$ matrix with row and column sums $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$. We first generalize the values $c_i$ and $r_j$. Index the rows and columns of $A$ by the elements of two disjoint sets $R = R(A)$ and $C = C(A)$. For all $i \in R \cup C$, set $w^1_A(i) = 1$; for all $s \geq 2$, $i \in R$, $j \in C$, set

$$w^s_A(i) = \sum_{k \in C} a_{ik} w^{s-1}_A(k), \quad w^s_A(j) = \sum_{k \in R} a_{kj} w^{s-1}_A(k).$$

Finally, for all $s \geq 1$, set

$$w^s_A(R) = \sum_{k \in R} w^s_A(k), \quad w^s_A(C) = \sum_{k \in C} w^s_A(k).$$

Note that $w^2_A(i) = r_i$ if $i \in R$, and $w^2_A(i) = c_i$ if $i \in C$. Also, if $A$ is the adjacency matrix of a graph, $w^s_A(i)$ is the number of walks on $s$ vertices starting with the vertex $i$.

Using somewhat different notation, in [5] it is proved that for every $m \times n$ matrix $A$ and all odd $p$ and $r$ such that $p > r \geq 1$,

$$\sigma^{p-r}(A) w^r_A(R) \geq w^p_A(R). \tag{2}$$

Moreover, for all $s \geq 1$,

$$\sigma^{2s}(A) = \lim_{r \to \infty} \frac{w^{2r+2s+1}_A(R)}{w^{2r+1}_A(R)} = \lim_{r \to \infty} \max_{k \in R(A)} \frac{w^{2r+2s+1}_A(k)}{w^{2r+1}_A(k)} \tag{3}$$

unless the eigenspace of $AA^*$ corresponding to $\sigma^2(A)$ is orthogonal to $j_m$.

Note that: (i) inequality (2) may not hold if $p$ or $r$ are even (see [1], p. 728 and [4], p. 262); (ii) equalities (3) hold if $A$ is a nonzero scalar matrix; (iii) inequality (2) implies a number of known results on the spectral radius of graphs (see [4], p. 258).

Inequality (2) can be proved using the Rayleigh principle. This simple approach helps produce other similar bounds of increasing complexity. We shall focus on the following general inequality.

**Theorem 1** Let $A$ be a matrix, $R = R(A)$ and $C = C(A)$. Then for all $r \geq 1$,

$$\sigma(A) \sqrt{\frac{w^r_A(R)}{w^r_A(C)}} \geq \left| \sum_{i \in R, j \in C} a_{ij} \sqrt{\frac{w^r_A(i)}{w^r_A(j)}} \right|. \tag{4}$$

Particularly, for $r = 1$ Theorem 1 reads as

$$\sigma(A) \geq |\Sigma(A)| / \sqrt{nm}. \tag{5}$$

Also, since $w^2_A(R) = w^2_A(C) = \Sigma(|A|)$, for $r = 2$ Theorem 1 extends inequality (1) to

$$\sigma(A) \Sigma(|A|) \geq \left| \sum_{i \in R, j \in C} a_{ij} \sqrt{\frac{w^2_A(i)}{w^2_A(j)}} \right|. \tag{6}$$

It is natural to study when equality holds in inequalities (2) and (4). To this end we first introduce some combinatorial concepts.
2.1 A few combinatorial concepts

For any matrix \( A = (a_{ij}) \), let \( B(A) \) be the bipartite graph with vertex classes \( R(A) \) and \( C(A) \) such that \( i \in R(A) \) is joined to \( j \in C(A) \) whenever \( a_{ij} \neq 0 \).

Call a matrix \( A \) **connected** if \( B(A) \) is connected. Note that a symmetric matrix is connected exactly when it is irreducible.

For scalar matrices connectedness can be expressed in terms of their powers.

**Proposition 2** A scalar matrix \( A \) is connected if and only if for every \( i \in R(A) \), \( j \in C(A) \), there exists \( r \) such that the \((i, j)\) entry of \((AA^*)^rA\) is nonzero.

Call a maximal connected submatrix of \( A \) a **component** of \( A \).

We say that \( A \) is **cogredient** to \( B \) if there exist permutation matrices \( P \) and \( Q \) such that \( A = PBQ \).

The following two assertions are obvious.

**Proposition 3** If a matrix \( A \) has no zero rows or columns, then it is cogredient to a block diagonal matrix

\[
\begin{pmatrix}
A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_r
\end{pmatrix}
\]

where \( A_1, \ldots, A_r \) are the components of \( A \).

**Proposition 4** The multiset of the nonzero singular values of \( A \) is the union of the multisets of the nonzero singular values of its components. In particular, for every matrix \( A \),

\[
\sigma(A) = \max \{\sigma(C) : C \text{ is a component of } A\}.
\]

Let \( A \) be a nonzero scalar matrix. We call \( A \) **regular** if its row sums are equal and so are its columns sums.

We call \( A \) **pseudo-regular** if \( w_3^A(i) = \lambda w_3^A(i) \) for all \( i \in R(A) \) and fixed \( \lambda \). Equivalently, \( A \) is pseudo-regular if the vector with coordinates \( w_3^A(i), i \in R(A) \) is an eigenvector of \( AA^* \).

If each component \( C \) of \( A \) is regular and \( \sigma(C) = \sigma(A) \), we call \( A \) **almost-regular**.


It is easy to see that regular matrices are almost regular, and that almost regular matrices are pseudo-regular. However the matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

is connected and pseudo-regular, but not regular. Note also that \( A^* \) is not pseudo-regular.

Here is a complete characterization of pseudo-regular matrices.
Proposition 5 A nonzero $m \times n$ scalar matrix $A$ is pseudo-regular if and only if the following conditions hold:

(i) the vector with entries $w_A^3(i), i \in R(A)$ is an eigenvector of $AA^*$ to some nonzero eigenvalue $\mu(AA^*)$;

(ii) the eigenvectors of $AA^*$ to every nonzero eigenvalue $\mu'(AA^*) \neq \mu(AA^*)$ are orthogonal to $j_m$.

Using this characterization, we can relax the definition of pseudo-regularity, preserving the same property scope.

Proposition 6 Suppose that $A$ is a scalar matrix, $r, s$ are odd, and $r > s \geq 3$. If $w_A^r(i) = \lambda w_A^s(i)$ for all $i \in R(A)$ and fixed $\lambda$, then $A$ is pseudo-regular.

2.2 Sufficient conditions for equality in (2) and (4)
The following theorem gives a condition for equality in (2).

Theorem 7 Suppose that $A$ is a scalar matrix with $R = R(A)$. If

\[
\sigma^{2s}(A) w_A^{2r+1}(R) = w_A^{2r+2s+1}(R),
\]

for some $s \geq 1, r \geq 0$, then $A$ is pseudo-regular.

Similar double condition implies a stronger conclusion.

Theorem 8 Suppose that $A$ is a scalar matrix with $R = R(A), C = C(A)$. If

\[
\begin{align*}
\sigma^{2s}(A) w_A^1(R) &= w_A^{2s+1}(R), \\
\sigma^{2r}(A) w_A^1(C) &= w_A^{2r+1}(C)
\end{align*}
\]

for some $r, s \geq 1$, then $A$ is almost regular.

Next, we generalize the second part of the aforementioned theorem of Hoffman, Wolfe, and Hofmeister giving conditions for equality in (4).

Theorem 9 Let $A = (a_{ij})$ be a scalar matrix and $r \geq 1, s \geq 1$. The following three conditions are equivalent:

(i) $A$ is almost regular;

(ii) $|w_A^r(i) w_A^r(j)| = \sigma(A)^2 |w_A^r(R) w_A^r(C)|$ whenever $a_{ij} \neq 0$;

(iii) we have

\[
\sigma(A) \sqrt{|w_A^r(R) w_A^r(C)|} = \sum_{i \in R, j \in C} |a_{ij} \sqrt{|w_A^r(i) w_A^r(j)|}|.
\]
A stronger condition holds for equality in (4) with \( r = 1 \).

**Theorem 10** A scalar matrix \( A \in M_{m,n} \) is regular if and only if \( \sigma (A) = |\Sigma (A)| / \sqrt{nm} \).

Note that the assumption that \( A \) is scalar is essential in Theorems 7 to 10. Indeed letting
\[
A = \begin{pmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{pmatrix},
\]
we see that
\[
\sigma (A) = 2, \quad \Sigma (A) = 4, \quad w^1_A (R) = w^1_A (C) = 2, \\
w^2_A (R) = w^2_A (C) = 4, \quad w^3_A (R) = 8,
\]
and
\[
\sum_{i \in R, j \in C} a_{ij} \sqrt{|w^2_A (i) w^2_A (j)|} = \sum_{i \in R, j \in C} 2a_{ij} = 8.
\]
Thus, we have
\[
\sigma^2 (A) w^3_A (R) = w^3_A (R),
\]
\[
\sigma (A) \sqrt{|w^2_A (R) w^2_A (C)|} = \sum_{i \in R, j \in C} a_{ij} \sqrt{|w^2_A (i) w^2_A (j)|},
\]
\[
\sigma (A) = |\Sigma (A)| / 2,
\]
although \( A \) is not scalar.

### 3 Proofs

**Proof of Theorem 1** Set \( x_i = \sqrt{w^r_{|A|} (i) / w^r_{|A|} (R)} \) for all \( i \in R \) and let \( x = (x_i) \).
Likewise, set \( y_i = \sqrt{w^r_{|A|} (i) / w^r_{|A|} (C)} \) for all \( i \in C \) and let \( y = (y_i) \). Since \( \|x\| = \|y\| = 1 \), by Schur’s lemma [7], we obtain
\[
\sigma (A) = \max_{\|u\| = \|v\| = 1} |\langle Au, v \rangle| \geq |\langle Ax, y \rangle|
\]
\[
= \frac{1}{\sqrt{w^r_{|A|} (R) w^r_{|A|} (C)}} \sum_{i \in R, j \in C} a_{ij} \sqrt{w^r_{|A|} (i) w^r_{|A|} (j)},
\]
completing the proof.

In the proofs below we shall assume that \( A \) is an \( m \times n \), nonzero, nonnegative matrix with \( R = R (A) \) and \( C = C (A) \); \( r_1, \ldots, r_m \); \( c_1, \ldots, c_n \) are its row and column sums, and \( \sigma = \sigma_1 \geq \cdots \geq \sigma_m \) are its singular values.
Let $AA^* = VDV^*$ be the unitary decomposition of $AA^*$; thus, the columns of $V$ are the unit eigenvectors to $\sigma_1^2, \ldots, \sigma_m^2$ and $D$ is the diagonal matrix with $\sigma_1^2, \ldots, \sigma_m^2$ along its main diagonal. Then for every $l \geq 0$,

$$w_{A}^{2l+1} (R) = \sum \left( (AA^*)^l \right) = \sum (VDV^*) = \sum c_i \sigma_i^{2l},$$

where $c_i = \left| \sum_{j \in [m]} v_{ji} \right|^2 \geq 0$ is independent of $l$.

Note also that for all $r, s \geq 0$,

$$\sum_{i \in R} w_{A}^{2r+1} (i) w_{A}^{2s+1} (i) = w_{A}^{2r+2s+1} (R). \quad (9)$$

We omit the easy proof by induction on $s$.

**Proof of Theorem 7** In the above notation we have

$$\sigma_{i}^{2s} \sum_{i \in [m]} c_i \sigma_i^{2r} = \sigma_{i}^{2s} w_{A}^{2r+1} (R) = w_{A}^{2r+2s+1} (R) = \sum_{i \in [m]} c_i \sigma_i^{2r+2s}.$$

Hence, if $0 < \sigma_i^2 < \sigma^2$, then $c_i = 0$. Therefore, for all $r > 1$, we have $w_{A}^{2r+1} (R) = C \sigma_r^2$, where $C$ is independent of $r$. Specifically,

$$w_{A}^{5} (R) = C \sigma^4, \quad w_{A}^{7} (R) = C \sigma^6, \quad w_{A}^{9} (R) = C \sigma^8.$$

Note the following instances of identity (9)

$$w_{A}^{5} (R) = \sum_{k \in R} \left( w_{A}^{3} (k) \right)^2, \quad w_{A}^{7} (R) = \sum_{k \in R} w_{A}^{5} (k) w_{A}^{3} (k), \quad w_{A}^{9} (R) = \sum_{k \in R} \left( w_{A}^{5} (k) \right)^2.$$

Hence, using the Cauchy-Schwarz inequality, we obtain

$$C \sigma^9 = w_{A}^{7} (R) = \sum_{k \in R} w_{A}^{5} (k) w_{A}^{3} (k) \leq \sqrt{\sum_{k \in R} \left( w_{A}^{5} (k) \right)^2} \sqrt{\sum_{k \in R} \left( w_{A}^{3} (k) \right)^2} \leq \sqrt{w_{A}^{9} (R) w_{A}^{5} (R)} = C \sigma^6.$$

We have equality in the Cauchy-Schwarz inequality; hence for each $k \in R$, $w_{A}^{5} (k) = \lambda w_{A}^{3} (k)$, where $\lambda$ is independent of $k$. Therefore $A$ is pseudo-regular, completing the proof. \(\square\)

**Proof of Theorem 8** In our proof we first show that $\sigma (A_i) = \sigma$ for every component of $A_i$ and that conditions (6) and (7) hold for each component of $A$. Let $A_1, \ldots, A_k$ be the components of $A$. For each $i \in [k]$, by inequality (2) we have

$$\sigma^{2r} (A_i) |R (A_i)| \geq w_{A_i}^{2r+1} (R (A_i)),$$

$$\sum_{i \in [k]} \left( w_{A_i}^{3} (k) \right)^2 \leq 2 \sqrt{\sum_{i \in [k]} \left( w_{A_i}^{5} (k) \right)^2} \sqrt{\sum_{i \in [k]} \left( w_{A_i}^{3} (k) \right)^2} \leq 2 \sqrt{w_{A}^{9} (R) w_{A}^{5} (R)} = 2 C \sigma^6.$$

We have equality in the Cauchy-Schwarz inequality; hence for each $k \in R$, $w_{A_i}^{5} (k) = \lambda w_{A_i}^{3} (k)$, where $\lambda$ is independent of $k$. Therefore $A_i$ is pseudo-regular, completing the proof. \(\square\)
and so
\[ \sigma^2 |R| \geq \sum_{i \in [k]} \sigma^2 (A_i) |R (A_i)| \geq \sum_{i \in [k]} w_{A_i}^{2r+1} (R (A_i)) = w_A^{2r+1} (R). \]

Therefore, condition (6) implies that \( \sigma (A_i) = \sigma \) for all \( i \in [k] \). We see also that condition (6), and likewise condition (7), holds for every component of \( A \); hence, we can assume that \( A \) is the sole component, i.e., \( A \) is connected. To finish the proof, we have to show that \( A \) is regular.

Since \( \sigma^2 w_A^{2s-1} (R) \geq w_A^{2s+1} (R) > 0 \) for all \( s \geq 1 \), condition (6) implies that
\[ \sigma^2 (A) |R| = w_A^3 (R) = \sum_{i \in R} w_A^3 (i) = \sum_{i, k \in R, j \in C} a_{ij} a_{kj} \]

Hence, \( j_m \) is an eigenvector of \( AA^* \) to \( \sigma^2 \) and so, for each \( i \in R \),
\[ \sigma^2 = \sum_{j \in C, k \in R} a_{ij} a_{kj} = \sum_{j \in C} a_{ij} c_j. \]

Let
\[ \delta_R = \min_{i \in R} r_i, \quad \delta_C = \min_{i \in C} c_i, \quad \Delta_R = \max_{i \in R} r_i, \quad \Delta_C = \max_{i \in C} c_i, \]

and select \( s \in R \) such that \( r_s = \delta_R \). Then
\[ \sigma^2 = \sum_{j \in C, k \in R} a_{sj} a_{kj} = \sum_{j \in C} a_{sj} c_j \leq \Delta_C \sum_{j \in C} a_{sj} = \Delta_C \delta_R. \] (10)

Likewise, we see that \( \sigma^2 \geq \delta_C \Delta_R \). Applying the same argument to \( A^* \) we find that
\[ \Delta_C \delta_R \leq \sigma^2 \leq \Delta_R \delta_C. \]

Therefore, \( \Delta_C \delta_R = \sigma^2 = \Delta_R \delta_C \). This implies that equality holds in (10), and so \( c_j = \Delta_C \) whenever \( a_{sj} \neq 0 \). Likewise, we see that if \( t \in R \) is such that \( c_t = \Delta_C \), then \( r_j = \delta_R \) whenever \( a_{jt} \neq 0 \). Since \( A \) is connected \( r_1 = \cdots = r_m \) and \( c_1 = \cdots = c_n \), completing the proof. \( \square \)

**Proof of Theorem 9** The implications \( (i) \implies (ii) \implies (iii) \) are obvious, so we shall focus on \( (iii) \implies (i) \). As in the proof of Theorem 8, we first reduce the argument to connected matrices. Let \( A_1, \ldots, A_k \) be the components of \( A \). By Theorem 1, for each \( s \in [k] \), we have
\[ \sigma (A_s) \sqrt{w_{A_s}^r (R (A_s)) w_{A_s}^r (C (A_s))} \geq \sum_{i \in R(A_s), j \in C(A_s)} a_{ij} \sqrt{w_{A_s}^r (i) w_{A_s}^r (j)}, \]
and, using the Cauchy-Schwarz inequality,
\[
\sigma \sqrt{\frac{w_A^r (R) w_A^r (C)}{A}} = \sigma \sqrt{\sum_{s \in [k]} w_{A_s}^r (R (A_s)) \sum_{s \in [k]} w_{A_s}^r (C (A_s))} \\
\geq \sigma \sum_{s \in [k]} \sqrt{w_{A_s}^r (R (A_s)) w_{A_s}^r (C (A_s))} \\
\geq \sum_{s \in [k]} \sigma (A_s) \sqrt{w_{A_s}^r (R (A_s)) w_{A_s}^r (C (A_s))} \\
\geq \sum_{s \in [k]|i \in R(A_s), j \in C(A_s)} \sum_{s \in [k]} a_{ij} \sqrt{w_{A_s}^r (i) w_{A_s}^r (j)} \\
= \sum_{i \in R, j \in C} a_{ij} \sqrt{w_A^r (i) w_A^r (j)}.
\]

Therefore, condition (8) implies that \( \sigma (A_i) = \sigma \) for all \( i \in [k] \). We see also that condition (8) holds for every component of \( A \); hence, we can assume that \( A \) is the sole component, i.e., \( A \) is connected. To finish the proof we must show that \( A \) is regular.

Let \( B = (b_{ij}) \) be defined as a block matrix
\[
B = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.
\]

It is known ([2], p. 418) that the positive eigenvalues of \( B \) are the nonzero singular values of \( A \). Set for convenience \( R = [m] \) and \( C = [m+1..m+n] \). By induction on \( r \) it is easy to see that for every \( r \geq 0 \) and for each \( i \in [m+n] \), the value \( w_A^{r+1} (i) \) is equal to the \( i \)th row sum of \( B^r \).

Let
\[
x_i = \sqrt{w_{[A]}^r (i)} \text{ for } i \in [m], \quad y_i = \sqrt{w_{[A]}^r (i + m)} \text{ for } i \in [n], \\
x = (x_1, \ldots, x_m), \quad y = (y_1, \ldots, y_n), \quad z = (x_1, \ldots, x_m, y_1, \ldots, y_n).
\]

Our main goal is to show that
\[
w_{[A]}^r (1) = \cdots = w_{[A]}^r (m), \quad w_{[A]}^r (m + 1) = \cdots = w_{[A]}^r (m + n)
\]

Equation (8) and the Rayleigh principle imply that \( z \) is an eigenvector of \( B \) to \( \sigma \); hence \( z \) is an eigenvector of \( B^r \) to \( \sigma^r \). Assume first that \( r \) is even, say \( r = 2k \). We have
\[
B^{2k-1} z = \begin{pmatrix} 0 & (AA^*)^{k-1} A \\ (A^*A)^{k-1} A^* & 0 \end{pmatrix} z = \sigma^{2k-1} z,
\]
and so,
\[
\sigma^{2k-1} x_i = \sum_{j \in [m+1]} b_{ij} y_{j-m} \text{ for } i \in [m], \quad (12)
\]
\[
\sigma^{2k-1} y_{i-m} = \sum_{j \in [m]} b_{ij} x_j \text{ for } i \in [m+1..n]. \quad (13)
\]
Let
\[ \delta_R = \min_{i \in [m]} w_A^{2k}(i), \quad \delta_C = \min_{i \in C} w_A^{2k}(i), \quad \Delta_R = \max_{i \in [R]} w_A^{2k}(i), \quad \Delta_C = \max_{i \in C} w_A^{2k}(i). \]

Select \( s \in [m] \) such that \( x_s = \delta_R \). Then, by (12),
\[ \sigma^{2k-1}_R \sqrt{\delta_R} = \sum_{j=m+1}^n b_{sj} y_{j-m} \leq \sqrt{\Delta_C} \sum_{j=m+1}^n b_{sj} = \sqrt{\Delta_C} \delta_R, \]
and so \( \sigma^{2k-1} \leq \sqrt{\Delta_C \delta_R} \). Likewise, we see that \( \sigma^{2k-1} \geq \sqrt{\Delta_R \delta_C} \). Applying the same argument to equation (13), we find that
\[ \sqrt{\Delta_C \delta_R} \leq \sigma^{2k-1} \leq \sqrt{\Delta_R \delta_C}. \]
Therefore, \( \Delta_C \delta_R = \sigma^{2k-1} = \Delta_R \delta_C \). This implies that equality holds in (14), and so \( w_A^{2k}(j) = \Delta_C \) whenever \( b_{sj} \neq 0 \). Likewise, we see that if \( t \in [m+1..n] \) is such that \( w_A^{2k}(t) = \Delta_C \), then \( w_A^{2k}(j) = \delta_R \) whenever \( b_{jt} \neq 0 \). Since \( A \) is connected, (11) holds for even \( r \). The proof of (11) for odd \( r \) goes along the same lines and we omit it.

Note that (11) implies that \( j_{m+n} \) is an eigenvector to \( B \) and thus all row and column sums of \( A \) are equal, completing the proof.

**Proof of Theorem 10** Suppose \( A \) is regular. Schur’s inequality [7]
\[ \sigma^2(A) \leq \max_{i \in R, j \in C} r_i c_j, \]
implies that \( \sigma(A) \leq \Sigma(A) / \sqrt{nm} \). In view of (5), we deduce that \( \sigma(A) = \Sigma(A) / \sqrt{nm} \).

Suppose now \( \sigma(A) = \Sigma(A) / \sqrt{nm} \). We have
\[ \sigma^2(A) m \geq \langle AA^* j_m, j_m \rangle = \sum_{i,k \in R, j \in C} a_{ij} a_{kj} = \sum_{j \in C} c_j^2 \geq \frac{1}{n} (\Sigma(A))^2 = \sigma^2(A) m, \]
implying that \( c_1 = \cdots = c_n \). Likewise we find that \( r_1 = \cdots = r_m \), completing the proof.

**Concluding remark**
It seems a challenging problem to investigate the cases of equality in (10) and (11) for arbitrary matrices.

**References**


