A spectral Erdős-Stone-Bollobás theorem

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Abstract
Let $r \geq 3$ and $(c/r^r) \log n \geq 1$. If $G$ is a graph of order $n$ and its largest eigenvalue $\mu(G)$ satisfies

$$\mu(G) \geq (1 - 1/(r - 1) + c) n,$$

then $G$ contains a complete $r$-partite subgraph with $r - 1$ parts of size $\lfloor (c/r^r)^r \log n \rfloor$ and one part of size greater than $n^{1-c^{-1}}$.

This result implies the Erdős-Stone-Bollobás theorem, the essential quantitative form of the Erdős-Stone theorem. Another easy consequence is that if $F_1, F_2, \ldots$ are $r$-chromatic graphs satisfying $v(F_n) = o(\log n)$, then

$$\lim_{n \to \infty} \frac{1}{n} \max \{ \mu(G) : v(G) = n \text{ and } F_n \not\subseteq G \} = 1 - \frac{1}{r - 1}.$$ 

Keywords: largest eigenvalue; $r$-partite subgraph; Erdős-Stone-Bollobás theorem.

This note is part of an ongoing project aiming to build extremal graph theory on spectral basis. Here we give a spectral version of the Erdős-Stone theorem.

Given $r \geq 3$ and $c > 0$, let $g(n, r, c)$ be the maximum integer such that every graph with $n$ vertices and $\lfloor (1 - 1/(r - 1) + c) n^2/2 \rfloor$ edges contains a complete $r$-partite graph with each part of size $g(n, r, c)$. The fundamental Erdős-Stone theorem [5] states that $g(n, r, c)$ tends to infinity with $n$. In [3] Bollobás and Erdős found that, in fact, $g(n, r, c) = \Theta(\log n)$. Below we give a spectral version of this result.

Our notation follows [2]; thus, $K_r(s_1, \ldots, s_r)$ denotes the complete $r$-partite graph with parts of sizes $s_1, \ldots, s_r$, and $\mu(G)$ stands for the spectral radius of the adjacency matrix of a graph $G$; log denotes the logarithm base $e$.

The main result of this note is the following theorem.

**Theorem 1** Let $r \geq 3$, $(c/r^r)^r \log n \geq 1$, and let $G$ be a graph with $n$ vertices. If

$$\mu(G) \geq (1 - 1/(r - 1) + c) n,$$

then $G$ contains a $K_r(s, \ldots, s, t)$ with $s \geq \lfloor (c/r^r)^r \log n \rfloor$ and $t > n^{1-c^{-1}}$. 

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We shall prove the following easy consequence of this result.

**Corollary 2** Let \( r \geq 3 \) and let \( F_1, F_2, \ldots \) be \( r \)-chromatic graphs satisfying \( v(F_n) = o\left(\log n\right) \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \max \{\mu(G) : v(G) = n \text{ and } F_n \not\subseteq G\} = 1 - \frac{1}{r-1} \tag{2}
\]

and

\[
\lim_{n \to \infty} \left(\frac{n}{2}\right)^{-1} \max \{e(G) : v(G) = n \text{ and } F_n \not\subseteq G\} = 1 - \frac{1}{r-1}. \tag{3}
\]

Before turning to the proofs of Theorem 1 and Corollary 2, some remarks seem in place.

1. The relation between \( c \) and \( n \) in Theorem 1 needs explanation. First, for fixed \( c \), it shows how large must be \( n \) so that the vertex classes of the required \( K_r(s, \ldots, s, t) \) are nonempty. But also \( c \) may depend on \( n \), e.g., letting \( c = 1/\log \log n \), the conclusion is meaningful for sufficiently large \( n \).

2. Note that, in Theorem 1, if condition (1) holds for some \( c \), the conclusion holds for every positive \( c' < c \) provided \( n \) is sufficiently large, i.e., as \( n \) grows, we can find a larger and more lopsided \( K_r(s, \ldots, s, t) \).

3. Since \( \mu(G) \geq 2e(G)/v(G) \), Theorem 1 implies the following form of the Erdős-Stone-Bollobás theorem:

Let \( r \geq 3 \), \( (c/r^r)^r \log n \geq 1 \), and let \( G \) be a graph with \( n \) vertices. If

\[ e(G) \geq \left(1 - 1/(r-1) + c\right)n^2/2, \]

then \( G \) contains a \( K_r(s, \ldots, s, t) \) with \( s \geq \lceil (c/r^r)^r \log n \rceil \) and \( t > n^{1-c^{-1}} \).

Other lower bounds on \( \mu(G) \), such as, e.g.,

\[ \mu^2(G) \geq \frac{1}{n} \sum_{u \in V(G)} d^2(u), \]

imply other new versions of this theorem.

4. Suppose that \( c \) is a sufficiently small positive constant. Choosing randomly a graph \( G \) of order \( n \) with \( \lceil (1 - 1/(r-1) + c)n^2/2 \rceil \) edges, we have \( \mu(G) \geq (1 - 1/(r-1) + c)n \), but \( G \) contains no \( K_2(\lceil C \log n \rceil, \lceil C \log n \rceil) \) for some \( C > 0 \), independent of \( n \). Hence, for constant \( c \) Theorem 1 is best possible up to a constant factor.

5. In the context of the project mentioned in the introduction, Corollary 2 solves asymptotically the following general extremal problem:

Given a family \( \mathcal{F} \) of forbidden subgraphs with chromatic number at least 3, find the maximum spectral radius of a graph of order \( n \) containing no member of \( \mathcal{F} \).

We turn now to the proofs of Theorem 1 and Corollary 2.

Write \( k_r(G) \) for the number of \( r \)-cliques of a graph \( G \). Our proof of Theorem 1 is based on the following results.
Theorem 3 ([4], Theorem 2) If \( r \geq 2 \) and \( G \) is a graph of order \( n \), then
\[
k_{r+1}(G) \geq \left( \frac{\mu(G)}{n} - 1 + \frac{1}{r} \right) \frac{r (r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1}.
\]

Theorem 4 ([7], Theorem 1) Let \( r \geq 2 \), \( \alpha^r \log n \geq 1 \), and let \( G \) be a graph of order \( n \). If \( k_r(G) \geq \alpha n^r \), then \( G \) contains a \( K_r(s, \ldots, s, t) \) with \( s = \lfloor \alpha^r \log n \rfloor \) and \( t > n^{1-\alpha r^{-1}} \).

Proof of Theorem 1 In view of \( \mu(G) \geq (1 - 1/(r-1) + c)n \), Theorem 3 implies that
\[
k_r(G) > c \frac{(r-1)(r-2)}{r} \left( \frac{n}{r-1} \right)^r > \frac{c}{r^r n^r}.
\]
Now, letting \( \alpha = c/r^r \), Theorem 4 implies that \( G \) contains a \( K_r(s, \ldots, s, t) \) with
\[
s \geq \lfloor \alpha^r \log n \rfloor = \lfloor (c/r^r)^r \log n \rfloor, \quad \text{and} \quad t > n^{1-\alpha r^{-1}} > n^{1-c^{-1}},
\]
completing the proof.

Proof of Corollary 2 Set \( c_n = r^r \left( v(F_n) / \log n \right)^{1/r} \) and let \( G \) be a graph of order \( n \) not containing \( F_n \). Then clearly \( G \) contains no \( K_r(s, \ldots, s) \) for \( s = v(F_n) \), and since
\[
(c_n/r^r)^r \log n = v(F_n) \geq 1.
\]
For \( n \) large enough, we have \( c_n < 1/2 \) and
\[
n^{1-c_n^{-1}} > (c_n/r^r)^r \log n = v(F_n),
\]
hence, Theorem 1 implies that
\[
\frac{\mu(G)}{n} \leq 1 - \frac{1}{r-1} + c_n.
\]
Thus, in view of
\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} r^r \left( \frac{v(F_n)}{\log n} \right)^{1/r} = r^r \left( \lim_{n \to \infty} \frac{v(F_n)}{\log n} \right)^{1/r} = 0,
\]
we obtain
\[
\limsup_{n \to \infty} \frac{1}{n} \max \left\{ \mu(G) : v(G) = n \text{ and } F_n \not\in G \right\} \leq 1 - \frac{1}{r-1}.
\]
On the other hand, writing \( T_s(n) \) for the \( s \)-partite Turán graph of order \( n \), we see that
\[
\frac{\mu(T_{r-1}(n))}{n} \geq \frac{\delta(T_{r-1}(n))}{n} \geq 1 - \frac{1}{r-1} - \frac{1}{n}.
\]
Since $T_{r-1}(n)$ is $(r - 1)$-partite, it contains no copy of $F_n$. Therefore,
\[
\liminf_{n \to \infty} \frac{1}{n} \max \{ \mu(G) : v(G) = n \text{ and } F_n \not\subseteq G \} \geq 1 - \frac{1}{r - 1},
\]
completing the proof of (2).

Now (3) follows since
\[
\left( \frac{n}{2} \right)^{-1} e(G) = \frac{2e(G)}{n(n-1)} \leq \frac{\mu(G)}{n-1} \leq \frac{\mu(G)}{n} + \frac{1}{n}
\]
and
\[
\left( \frac{n}{2} \right)^{-1} e(T_{r-1}(n)) \geq \left( \frac{n}{2} \right)^{-1} \frac{n\delta(T_{r-1}(n))}{2} = \frac{\delta(T_{r-1}(n))}{n-1} \geq 1 - \frac{1}{r - 1}.
\]

\[\square\]

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References