Turán’s theorem inverted

Vladimir Nikiforov

Department of Mathematical Sciences, University of Memphis, Memphis TN 38152
e-mail: vnikifrv@memphis.edu

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Abstract

In this note we solve an open problem of Erdős from 1963 strengthening the fundamental theorem of Turán in extremal graph theory.

Let $K^r_+(s_1, \ldots, s_r)$ be the complete $r$-partite graph with classes of size $s_1 \geq 2, s_2, \ldots, s_r$ with an edge added to the first class. Letting $t_r(n)$ be the number of edges of the $r$-partite Turán graph of order $n$, we prove that:

For all $r \geq 2$ and all sufficiently small $c > 0$, every graph of sufficiently large order $n$ with $t_r(n) + 1$ edges contains a $K^r_+(\lfloor cn \ln n \rfloor, \ldots, \lfloor cn \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$.

We also give a corresponding stability theorem and two supporting results of wider scope.

Keywords: clique; $r$-partite subgraph; stability, Turán’s theorem

1 Introduction

This note is part of an ongoing project aiming to improve some classical results in extremal graph theory, see, e.g., [3], [7, 10]. Here we complete an investigation started by Erdős in 1963.

Let $t_r(n)$ be the number of edges of the $r$-partite Turán graph of order $n$. The fundamental Turán theorem implies that every graph on $n$ vertices with $t_r(n) + 1$ edges contains a $K_{r+1}$, the complete graph of order $r + 1$. Thus, it is natural to ask:

Which supergraphs of $K_{r+1}$ are present in graphs on $n$ vertices with $t_r(n) + 1$ edges?

Let $K^r_+(s_1, \ldots, s_r)$ be the complete $r$-partite graph with classes of size $s_1 \geq 2, s_2, \ldots, s_r$ with an edge added to the first class. An answer to the above question was stated by Erdős in [4] and proved in [6], Theorem 1:

Let $r \geq 2$ and $s \geq 2$. Then every graph of sufficiently large order $n$ with $t_r(n) + 1$ edges contains a $K^r_+(s, \ldots, s)$.

For $r = 2$, Erdős [4] gave a stronger result:

For all sufficiently small $\varepsilon > 0$, every graph of sufficiently large order $n$ with $t_2(n) + 1$ edges contains a $K^2_+(\lfloor cn \ln n \rfloor, \lfloor n^{1-\varepsilon} \rfloor)$ for some $c > 0$, independent of $n$. 

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Theorem 1 Let \( r \geq 2, \ 2/\ln n \leq c \leq r^{-(r+7)(r+1)}, \) and \( G \) be a graph of order \( n \). If \( G \) has \( t_r(n) + 1 \) edges, then \( G \) contains a \( K_{r}^{+}\left([c\ln n], \ldots, [c\ln n], \left[n^{1-\sqrt{c}}\right]\right) \).

For readers’ sake we present an immediate consequence of this assertion.

Corollary 2 Let \( r \geq 2, \ c = r^{-(r+7)(r+1)}, \ n \geq e^{2/c}, \) and \( G \) be a graph of order \( n \). If \( G \) has \( t_r(n) + 1 \) edges, then \( G \) contains a \( K_{r}^{+}\left([c\ln n], \ldots, [c\ln n]\right) \).

Both Theorem 1 and Corollary 2 generalize Turán’s theorem since \( K_{r}^{+}\left([c\ln n], \ldots, [c\ln n]\right) \) contains a \( K_{r+1} \). Moreover, just like the Turán theorem can be complemented by a stability theorem (see, e.g., [5], [8], and [11]), we have stability results corresponding to Theorem 1 and Corollary 2.

Theorem 3 Let
\[
r \geq 2, \quad 2/\ln n \leq c \leq r^{-(r+7)(r+1)/2}, \quad 0 < \alpha < r^{-8/8},
\]
and let \( G \) be a graph of order \( n \). If \( G \) has \( \lceil(1 - 1/r - \alpha)n^2/2\rceil \) edges, then one of the following statements holds:

(i) \( G \) contains a \( K_{r}^{+}\left([c\ln n], \ldots, [c\ln n], \left[n^{1-2\sqrt{c}}\right]\right) \);

(ii) \( G \) contains an induced \( r \)-partite subgraph \( G_0 \) of order at least \( (1 - \sqrt{2\alpha})n \) and with minimum degree \( \delta(G_0) > (1 - 1/r - 2\sqrt{2\alpha})n \).

Here is a simplified version of Theorem 3, corresponding to Corollary 2:

Corollary 4 Let
\[
r \geq 2, \quad c = r^{-(r+7)(r+1)/2}, \quad 0 < \alpha < r^{-8/8}, \quad n \geq e^{2/c},
\]
and let \( G \) be a graph of order \( n \). If \( G \) has \( \lceil(1 - 1/r - \alpha)n^2/2\rceil \) edges, then one of the following statements holds:

(i) \( G \) contains a \( K_{r}^{+}\left([c\ln n], \ldots, [c\ln n]\right) \);

(ii) \( G \) contains an induced \( r \)-partite subgraph \( G_0 \) of order at least \( (1 - \sqrt{2\alpha})n \) and with minimum degree \( \delta(G_0) > (1 - 1/r - 2\sqrt{2\alpha})n \).

In our proofs we use some tools developed elsewhere. However, a crucial role is played also by the following two versatile statements, which, in turn, may have applications outside of the present note.

Lemma 5 Let \( 0 < \alpha \leq 1, \ 1 \leq c\ln n \leq \alpha m/2 + 1, \) and let \( F \) be a bipartite graph with parts \( A \) and \( B \) of size \( m \) and \( n \). If \( e(F) \geq \alpha mn \), then \( F \) contains a \( K_2(s,t) \) with parts \( S \subset A \) and \( T \subset B \) such that \(|S| = [c\ln n]\) and \(|T| = t > n^{1-c\ln \alpha/2}\).

Theorem 6 Let \( r \geq 2, \ 2/\ln n \leq c \leq r^{-(r+8)r}, \) and \( G \) be a graph \( G \) of order \( n \). If \( G \) contains a \( K_{r+1} \) and has minimum degree \( \delta(G) > (1 - 1/r - 1/r^4)n \), then \( G \) contains a
\[
K_{r}^{+}\left([c\ln n], \ldots, [c\ln n], \left[n^{1-\alpha^3}\right]\right).
\]
Remarks

- The relations between \( c \) and \( n \) in Theorems 1 and 3 need some explanation. First, for fixed \( c \), they show how large must be \( n \) to get valid conclusions. But, in fact, the relations are subtler, for \( c \) itself may depend on \( n \), e.g., letting \( c = 1/\ln \ln n \), the conclusions are meaningful for sufficiently large \( n \).

- Note that, in Theorems 1 and 3, if the conclusion holds for some \( c \), it holds also for \( 0 < c' < c \), provided \( n \) is sufficiently large. This implies the results of Erdős mentioned above.

- The stability conditions in Theorem 3 and Corollary 4 are stronger than the conditions in the stability theorems of [5], [8], and [11]. Indeed, condition (ii) implies that \( G_0 \) is an induced, almost balanced, and almost complete \( r \)-partite graph containing almost all the vertices of \( G \);

- The exponents \( 1 - \sqrt{c} \) and \( 1 - 2\sqrt{c} \) in Theorems 1 and 3 are far from the best ones, but are simple.

The next section contains notation and results needed to prove the theorems. The proofs are presented in Section 3.

2 Preliminary results

Our notation follows [2]; thus, given a graph \( G \), we write:

- \( V(G) \) for the vertex set of \( G \) and \( |G| \) for \( |V(G)| \);
- \( E(G) \) for the edge set of \( G \) and \( e(G) \) for \( |E(G)| \);
- \( \Gamma(u) \) for the set of neighbors of a vertex \( u \) and \( d(u) \) for \( |\Gamma(u)| \);
- \( \delta(G) \) for the minimum degree of \( G \);
- \( G[U] \) for the subgraph of \( G \) induced by a set \( U \subset V(G) \);
- \( H + u \) for \( G[V(H) \cup \{u\}] \), where \( H \subset G \) is a subgraph and \( u \in V(G) \);
- \( K_r(G) \) for the set of \( r \)-cliques of \( G \) and \( k_r(G) \) for \( |K_r(G)| \);
- \( K_s(M) \) for the set of \( s \)-cliques contained in members of a set \( M \subset K_r(G) \);
- \( K_r(s_1, \ldots, s_r) \) for the complete \( r \)-partite graph with parts of size \( s_1, \ldots, s_r \).

An \( r \)-joint of size \( t \) is the union of \( t \) distinct \( r \)-cliques sharing an edge. Write \( js_r(G) \) for the maximum size of an \( r \)-joint in \( G \).

Given a set \( M \subset K_r(G) \) and a subgraph \( H \subset G \) such that \( H = K_r(s_1, \ldots, s_r) \), we say that \( M \) covers \( H \) if \( E(H) \subset K_2(M) \) and \( H \) contains \( \min\{s_1, \ldots, s_r\} \) disjoint members of \( M \).

For our proofs we need the following facts, all obtained recently as tools for the project mentioned in the introduction.

Fact 7 ([3], Lemma 1) Let \( r \geq 2 \) and \( c \geq 0 \), and \( G \) be a graph of order \( n \). If

\[
e(G) > (1 - 1/r + c) n^2/2,
\]
then

\[ k_{r+1}(G) > c \left( \frac{r^2}{r+1} \left( \frac{n}{r} \right)^{r+1} \right). \]

\[ \square \]

**Fact 8 ([3], Lemma 6)** Let \( r \geq 2 \), and \( G \) be a graph of order \( n \). If \( G \) contains a \( K_{r+1} \) and \( \delta(G) > (1 - 1/r - 1/r^4)n \), then \( j_{s_{r+1}}(G) > n^{r-1/r + 3} \).

\[ \square \]

**Fact 9 ([3], Theorem 7)** Let \( r \geq 2 \), \( n > r^8 \), and \( G \) be a graph of order \( n \). If \( e(G) > t_r(n) \), then \( G \) has an induced subgraph \( G' \) of order \( n' > (1 - 1/r - 1/r^4)n \) such that either

\[ e(G') > \left( \frac{r - 1}{2r} + \frac{1}{r^4(r^2 - 1)} \right)(n')^2 \tag{1} \]

or

\[ K_{r+1} \subset G', \quad \text{and} \quad \delta(G') > (1 - 1/r - 1/r^4)n'. \tag{2} \]

\[ \square \]

**Fact 10 ([7], Theorem 1)** Let \( r \geq 2 \), \( \alpha r \ln n \geq 1 \), and \( G \) be a graph of order \( n \). Every set \( M \subset K_r(G) \) satisfying \( |M| \geq \alpha n \) covers a \( K_r(s, \ldots, s, t) \) with \( s = \lfloor \alpha r \ln n \rfloor \) and \( t > n^{1 - \alpha r^{-1}} \).

\[ \square \]

3 Proofs

**Proof of Lemma 5** Set \( s = \lfloor c \ln n \rfloor \) and let

\[ t = \max \{ x : \text{there exists } K_2(s, x) \subset F \text{ with part of size } s \text{ in } A \}. \]

Thus \( d(X) \leq t \) for each \( X \subset A \) with \( |X| = s \), and so,

\[ t \binom{m}{s} \geq \sum_{X \subset A, |X| = s} d(X) = \sum_{u \in B} \binom{d(u)}{s}. \tag{3} \]

Setting

\[ f(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s - 1 \\ 0 & \text{if } x < s - 1, \end{cases} \]

and noting that \( f(x) \) is a convex function, we find that,

\[ \sum_{u \in B} \binom{d(u)}{s} = \sum_{u \in B} f(d(u)) \geq nf \left( \frac{1}{n} \sum_{u \in B} d(u) \right) = n \binom{e(F)/n}{s} \geq n \binom{\alpha m}{s}. \]

Combining this inequality with (3) and rearranging, we find that

\[ t \geq n \alpha m (am - 1) \ldots (am - s) \ldots (m - s + 1) > n \left( \frac{am - s + 1}{m} \right)^s \geq n \left( \frac{\alpha}{2} \right)^s \geq n^{1+c \ln(\alpha/2)}, \]

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completing the proof.

\[ \square \]

**Proof of Theorem 6** Let \( r, c, n, \) and the graph \( G \) satisfy the conditions of the theorem. Note first that for every \( R \in K_{r-1}(G) \),

\[
d(R) = \left| \bigcap_{u \in R} \Gamma(u) \right| \geq \sum_{u \in R} d(u) - (r - 2) n \geq (r - 1) \delta(G) - (r - 2) n > \frac{n}{r^2}.
\]

Thus, there exists an edge \( uv \) implying that \( |Y| - |\Gamma(u) \cap \Gamma(v) \cap V(G)| \). We claim that there exists \( k_{r-1}(G[B]) > n^{r-1}/r^{r+4} \).

Define the set \( X \) as

\[
X = \{ R : R \in K_r(G) \text{ and } |R \cap B| \geq r - 1 \}.
\]

In view of (4) and (5), we find that

\[
|X| \geq \frac{1}{r} \sum_{P \in K_{r-1}(G[B])} d(P) > \frac{1}{r} \times \frac{n}{r^2} \times \frac{n^{r-1}}{r^{r+4}} = \frac{n^r}{r^{r+7}}.
\]

For a set \( N \subseteq K_r(G) \) and a clique \( R \in K_{r-1}(N) \) let \( d_N(R) \) be the number of members of \( N \) containing \( R \). We claim that there exists \( Y \subseteq X \) with \( |Y| > n^r/r^{r+8} \) such that \( d_Y(R) > n/r^{r+8} \) for all \( R \in K_{r-1}(Y) \). Indeed, set \( Y = X \) and apply the following procedure:

**While** there exists an \( R \in K_{r-1}(Y) \) with \( d_Y(R) \leq n/r^{r+8} \) **do**

Remove from \( Y \) all \( r \)-cliques containing \( R \).

When the procedure stops, \( d_Y(R) > n/r^{r+8} \) for all \( R \in K_{r-1}(Y) \), and

\[
|X| - |Y| \leq |K_{r-1}(X)| \frac{n}{r^{r+8}} \leq \left( \frac{n}{r^2} \right) \frac{n^{r-1}}{r^{r+8}} < \frac{1}{r^{r+7}} n^r,
\]

implying that \( |Y| > n^r/r^{r+8} \), as claimed.

Since

\[
|K_{r-1}(Y)| \geq r |Y| / n > r \times r^{-r-8} n^r / n = n^{r-1}/r^{r+7},
\]

by Fact 10, \( K_{r-1}(Y) \) covers a subgraph \( H = K_{r-1}(m, \ldots, m) \) with \( m = \left\lfloor r^{-(r+7)(r-1)} \ln n \right\rfloor. \)

Select a set \( A \) of \( m \) disjoint \((r-1)\)-cliques in \( H \) that are members of \( K_{r-1}(Y) \) and define a bipartite graph \( F \) with parts \( A \) and \( B \), joining \( R \in A \) to \( v \in B \) if \( R + v \in Y \).
Let $\alpha = 1/r^{r+8}$ and set $s = \lfloor c \ln n \rfloor$. Since

$$dy(R) > \frac{1}{r^{r+8}} n \geq \alpha n$$

for all $R \in K_{r-1}(Y)$, we have $e(F) > \alpha mn$. Also, we find that

$$s \leq c \ln n \leq \frac{1}{r^{r+8}} r \ln n \leq \frac{1}{2r^{r+8}} \times \frac{1}{r(r+7)(r-1)} \ln n \leq \frac{\alpha}{2} m + 1.$$

Hence, by Fact 5, $H$ contains a $K_2(s, t)$ with parts $S \subset A$ and $T \subset B$ such that $|S| = s$ and $|T| = t > n^{1-c \ln \alpha/2}$. A routine calculation shows that for $r \geq 2$,

$$\ln \alpha/2 = \ln \frac{1}{2r^{r+8}} \geq -r^3,$$

and so, $t > n^{1-cr^3}$.

Letting $H^*$ be the subgraph of $H$ induced by the union of the members of $S$, we see that $H^* = K_{r-1}(s, \ldots, s)$. Since $R + v \in Y$ for all $v \in T$ and $R \in K_{r-1}(H^*)$, we see that $Y$ covers a $K_r(s, \ldots, s, t)$. Note that at least $(r - 2)$ of the parts of $H^*$ belong to $B$, for otherwise we can select an $(r - 1)$-clique $Q$ in $H^*$ with two vertices outside $B$, and so, every $R \in Y$ containing $Q$ has two vertices outside $B$. This is a contradiction since $Y \subseteq X$ and all members of $X$ intersect $B$ in at least $r - 1$ vertices.

Let $H_1, \ldots, H_{r-1}$ be the parts of $H^*$, and assume by symmetry that $H_i \subset B$ for $i = 2, \ldots, r - 1$. Remove two vertices from $H_1$, add $u$ and $v$ to $H_1$, and write $H_1'$ for the resulting set. Clearly the sets $H_1', H_2, \ldots, H_{r-1}, T$ induce a subgraph containing a $K_r^+(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-cr^3} \rceil)$, completing the proof.

**Proof of Theorem 1** Let $G$ be a graph of order $n$ with $t_r(n) + 1$ edges. Fact 9 implies that there exists an induced subgraph $G' \subset G$ of order $n' > (1 - 1/r^2)n$ such that either (1) or (2) holds.

Assume first that $G'$ satisfies condition (1). Fact 7 implies that

$$k_{r+1}(G) \geq k_{r+1}(G') \geq \frac{2}{r^4(r^2 - 1)} \times \frac{r^2}{r + 1} \times \binom{n'}{r}^{r+1}$$

$$> \frac{2}{r^2(r^2 - 1)(r + 1)} \times \left(1 - \frac{1}{r^2}\right)^{r+1} \times \binom{n}{r}^{r+1}$$

$$> \frac{2}{r^2(r^2 - 1)(r + 1)} \times \left(1 - \frac{r + 1}{r^2}\right) \times \binom{n}{r}^{r+1}$$

$$> \frac{2(r^2 - r - 1)}{r^4(r^2 - 1)(r + 1)} \times \left(\binom{n}{r}^{r+1} > \frac{1}{r^{r+7}} n^{r+1} > c^{1/(r+1)} n^{r+1}.\right.$$}

Hence, by Fact 10, $G$ contains a $K_{r+1}(s, \ldots, s, t)$ with $s = \lfloor c \ln n \rfloor$ and

$$t > n^{1-c^{r/(r+1)}} > n^{1-c^{r}}.$$
Then, obviously, \( G \) contains a \( K_r^+ \left( \left[ c \ln n \right], \ldots, \left[ c \ln n \right], \left[ n^{1-\sqrt{c}} \right] \right) \), completing the proof.

Finally, assume that \( G' \) satisfies condition (2). Applying Theorem 6, we see that \( G' \) contains a

\[
K_r^+ \left( \left[ 2c \ln n' \right], \ldots, \left[ 2c \ln n' \right], \left( n' \right)^{1-2cr^3} \right).
\]

To complete the proof, note that

\[
2c \ln n' \geq 2c \ln \left( 1 - \frac{1}{r^2} \right) n \geq 2 \ln \left( 1 - \frac{1}{r^2} \right) + 2 \ln n \geq c \ln n
\]

and

\[
(n')^{1-2cr^3} \geq \left( 1 - \frac{1}{r^2} \right)^{1-2cr^3} n^{1-2cr^3} \geq \left( 1 - \frac{1}{r^2} \right) n^{1-2cr^3} > n^{1-\sqrt{c}}.
\]

\[\square\]

**Proof of Theorem 3** Let \( G \) be a graph of order \( n \) with \( e(G) > (1 - 1/r - \alpha) n^2/2 \). Set \( V = V(G), \varepsilon = \sqrt{2\alpha} \), and define the set \( M_\varepsilon \) as

\[
M_\varepsilon = \{ u \in V(G) : d(u) \leq (1 - 1/r - \varepsilon) n \}.
\]

Assume that condition (i) fails. We shall show that: (a) \( |M_\varepsilon| < \varepsilon n \); (b) the graph \( G_0 = G[V \setminus M_\varepsilon] \) satisfies condition (ii).

(a) The set \( M_\varepsilon \) satisfies \( |M_\varepsilon| < \varepsilon n \)

Assume for a contradiction that \( |M_\varepsilon| \geq \varepsilon n \), select \( M' \subset M_\varepsilon \) with

\[
|M'| = \lceil \varepsilon n \rceil
\]

and note that \( M' \) is nonempty since \( \varepsilon n = \sqrt{2\alpha} n > 1 \). Letting \( G' = G[V \setminus M'] \), we see that

\[
e(G) = e(G') + e(M', V \setminus M') + e(M') \leq e(G') + \sum_{u \in M'} d(u)
\]

\[
\leq e(G') + |M'| (1 - 1/r - \varepsilon) n.
\]

Assume for a contradiction that

\[
e(G') > \frac{r-1}{2r} \left( n - |M'| \right)^2
\]

and set \( p = n - |M'| \). In view of (6), we have

\[
p \geq n - \varepsilon n = \left( 1 - \sqrt{2\alpha} \right) n.
\]
Hence, by Theorem 1, $G$ contains a $K^+_{r'} \left( \lfloor 2c \ln p \rfloor, \ldots, \lfloor 2c \ln p \rfloor, \left[ p^{1-\sqrt{2c}} \right] \right)$. Since
\[ 2c \ln p \geq 2c \ln \left( 1 - \sqrt{2\alpha} \right) n \geq 2c \ln \left( 1 - \frac{1}{4r^4} \right) n \geq c \ln n \]
and
\[ p^{1-\sqrt{2c}} \geq \left( 1 - \sqrt{2\alpha} \right)^{1-\sqrt{2c}} n^{1-\sqrt{2c}} > \left( 1 - \sqrt{2\alpha} \right)n^{1-\sqrt{2c}} > n^{1-2\sqrt{c}}, \]
this contradicts the assumption that (i) fails.

Hereafter, we assume that
\[ e (G') \leq \frac{r - 1}{2r} (n - |M'|)^2. \]

From
\[ e (G') \geq e (G) - \sum_{u \in M} d (u) \geq (1 - 1/r - \alpha) n^2/2 - |M'| (1 - 1/r - \varepsilon) n, \]
we obtain
\[ \frac{r - 1}{2r} (n - |M'|)^2 \geq \left( \frac{r - 1}{r} - \alpha \right) \frac{n^2}{2} - |M'| \left( \frac{r - 1}{r} - \varepsilon \right) n. \]

After some algebra, we find that
\[ |M'| < \left( \varepsilon - \sqrt{\varepsilon^2 - \alpha} \right) n = \varepsilon \left( 1 - \sqrt{1/2} \right) n \]
or
\[ |M'| > \left( \varepsilon + \sqrt{\varepsilon^2 - \alpha} \right) n = \varepsilon \left( 1 + \sqrt{1/2} \right) n, \]
contradicting (6) in view of $\varepsilon \sqrt{1/2} n = \sqrt{2\alpha n} > \sqrt{2}$. Therefore, $|M_e| < \varepsilon n$.

(b) The graph $G_0 = G [V \setminus M_e]$ satisfies condition (ii).

By our choice of $M_e$, for every $u \in V \setminus M_e$ we have $d (u) > (1 - 1/r - \varepsilon) n$; thus
\[ \delta (G_0) > (1 - 1/r - \varepsilon) n - |M_e| > (1 - 1/r - 2\varepsilon) n = \left( 1 - 1/r - 2\sqrt{2\alpha} \right) n, \]
and so, $\delta (G_0)$ satisfies the required condition. All that remains to prove is that $G_0$ is $r$-partite.

If $G_0$ contains a $K_{r+1}$, in view of
\[ \delta (G_0) > \left( 1 - 1/r - 2\sqrt{2\alpha} \right) n > \left( 1 - 1/r - 1/r^4 \right) n, \]
using Theorem 6 as in the proof of Theorem 1, we see that $G$ contains a
\[ K^+_{r'} \left( \lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left[ n^{1-\sqrt{c}} \right] \right), \]
contradicting our assumption. Thus, $G_0$ is $K_{r+1}$-free. In view of
\[ \delta (G_0) > \left( 1 - 1/r - 1/r^4 \right) n > \left( 1 - \frac{3}{3r - 1} \right) |G_0|, \]
the theorem of Andrásfai, Erdős and Sós [1] implies that $G_0$ is $r$-partite, completing the proof. \]

We omit the proofs of Corollaries 2 and 4, since they are easy consequences of Theorem 1 and 3.
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References


