A spectral condition for odd cycles in graphs

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Abstract

Let \( G \) be a graph of sufficiently large order \( n \), and let the largest eigenvalue \( \mu(G) \) of its adjacency matrix satisfies \( \mu(G) > \sqrt{\frac{n^2}{4}} \). Then \( G \) contains a cycle of length \( t \) for every \( t \leq n/320 \).

This condition is sharp: the complete bipartite graph \( T_2(n) \) with parts of size \( \lfloor n/2 \rfloor \) and \( \lceil n/2 \rceil \) contains no odd cycles and its largest eigenvalue is equal to \( \sqrt{\frac{n^2}{4}} \).

This condition is stable: if \( \mu(G) \) is close to \( \sqrt{\frac{n^2}{4}} \) and \( G \) fails to contain a cycle of length \( t \) for some \( t \leq n/321 \), then \( G \) resembles \( T_2(n) \).

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Introduction

This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [3] and [6, 13].

It is known ([9], [14]) that if \( G \) is a graph of order \( n \) and the largest eigenvalue \( \mu(G) \) of its adjacency matrix satisfies \( \mu(G) > \sqrt{\frac{n^2}{4}} \), then a triangle exists in \( G \).

Here we show that the same premises imply the existence of other cycles as well.

Theorem 1 Let \( G \) be a graph of sufficiently large order \( n \) with \( \mu(G) > \sqrt{\frac{n^2}{4}} \). Then \( G \) contains a cycle of length \( t \) for every \( t \leq n/320 \).

Write \( T_2(n) \) for the complete bipartite graph with parts of size \( \lfloor n/2 \rfloor \) and \( \lceil n/2 \rceil \). Note that \( T_2(n) \) contains no odd cycles and \( \mu(T_2(n)) = \sqrt{\frac{n^2}{4}} \); thus, Theorem 1 gives a sharp spectral condition for the existence of odd cycles.

Moreover, there is stability in this condition: if \( \mu(G) \) is close to \( \sqrt{\frac{n^2}{4}} \) and \( G \) fails to contain a cycle of length \( t \) for some \( t \leq n/321 \), then \( G \) resembles \( T_2(n) \). Here is a precise form of this statement.
**Theorem 2** Let $0 < \theta < 2^{-16}$ and $n$ be sufficiently large. For every graph $G$ of order $n$ with $\mu (G) > (1/2 - \theta)n$, one of the following conditions holds:

(i) $G$ contains a cycle of length $t$ for every $t \leq n/321$;

(ii) there exists an induced bipartite subgraph $G_0 \subset G$ satisfying $|G_0| > (1 - 4\theta^{1/3})n$ and $\delta (G_0) > (1/2 - 7\theta^{1/3})n$.

The proofs of Theorems 1 and 2 are based on three results of independent interest.

**Lemma 3** Let $G$ be a graph of order $n$ with minimum degree $\delta$ and $\mu (G) = \mu$. If $(x_1, \ldots, x_n)$ is a unit eigenvector to $\mu$, then

$$\min \{x_1, \ldots, x_n\} \leq \frac{\delta}{\mu^2 + \delta n - \delta^2}.$$

**Lemma 4** Let $G$ be a graph of order $n$ with $\mu (G) = \mu$. If $(x_1, \ldots, x_n)$ is a unit eigenvector to $\mu$ and $u$ is a vertex satisfying $x_u = \min \{x_1, \ldots, x_n\}$, then

$$\frac{\mu (G - u)}{n - 1} > \frac{\mu (G)}{n} \left(1 + \frac{1}{n - 1} \left(1 - nx_u^2 - \frac{1}{n - 1}\right)\right).$$

Combining these two lemmas, we get Theorem 5 below. We hope that this technical statement can be used in other spectral extremal problems.

**Theorem 5** Let $0 < 4\alpha \leq 1$, $0 < 2\beta \leq 1$, $1/2 - \alpha/4 \leq \gamma < 1$, $K \geq 0$, and $n \geq (42K + 4)/\alpha^2\beta$. If $G$ is a graph of order $n$ with

$$\mu (G) > \gamma n - K/n \quad \text{and} \quad \delta (G) \leq (\gamma - \alpha)n,$$

then there exists an induced subgraph $H \subset G$ with $|H| \geq (1 - \beta)n$, satisfying one of the following conditions:

(i) $\mu (H) > \gamma (1 + \beta \alpha/2)|H|$;

(ii) $\mu (H) > \gamma |H|$ and $\delta (H) > (\gamma - \alpha)|H|$.

**Proofs**

We start with some notation and results needed for our proofs.

Our graph-theoretical notation follows [2]. Specifically, given a graph $G$, we write:

- $|G|$ for the number of vertices of $G$;
- $E (G)$ for the edge set of $G$;
- $k_3 (G)$ for the number of triangles of $G$;
- $d (u)$ for the degree of a vertex $u$;
- $\Gamma (u)$ for the set of neighbors of a vertex $u$;
- $\delta (G)$ for the minimum degree of $G$.

The following fact is a reduced version of Theorem 1 of [5].
Fact 6 Let $G$ be a nonbipartite graph of sufficiently large order $n$, and let $\delta (G) \geq n/3$. Then $C_t \subset G$ for every integer $t \in [4, \delta (G) + 1]$.

The following facts are particular cases of Theorems 2 and 4 in [3].

Fact 7 If $G$ is a graph of order $n$, then $k_3 (G) \geq (\mu (G) / n - 1/2) n^3/12$.

Fact 8 Let $0 < \theta \leq 2^{-16}$ and let $G$ be a triangle-free graph of order $n$ with $\mu (G) \geq (1/2 - \theta) n$. Then there exists an induced bipartite graph $H \subset G$ satisfying $|H| > (1 - 3\theta^{1/3}) n$ and $\delta (H) > (1/2 - 6\theta^{1/3}) n$.

Proof of Lemma 3

Set $\sigma = \min \{x_1, \ldots, x_n\}$. If $\sigma = 0$, the assertion holds trivially, so we assume that $\sigma > 0$. This implies also that $\delta > 0$. Taking $u \in V (G)$ to satisfy $d (u) = \delta$, we have

$$
\mu^2 \sigma^2 \leq \mu^2 x_u^2 = \left( \sum_{i \in \Gamma (u)} x_i \right)^2 \leq \delta \sum_{i \in \Gamma (u)} x_i^2 \leq \delta \left( 1 - \sum_{i \in V (G) \setminus \Gamma (u)} x_i^2 \right) \\
\leq \delta \left( 1 - (n - \delta) \sigma^2 \right) = \delta - (\delta n - \delta^2) \sigma^2,
$$

implying that $(\mu^2 + \delta n - \delta^2) \sigma^2 \leq \delta$, and the desired inequality follows.

Proof of Lemma 4

Set for short $c = 1 - nx_u^2$ and $\mu = \mu (G)$. We have

$$
\mu x_u = \sum_{v \in \Gamma (u)} x_v \quad \text{and} \quad \mu = 2 \sum_{vw \in E (G)} x_v x_w.
$$

Hence, by Rayleigh’s principle, we obtain

$$
\mu = 2 \sum_{vw \in E (G-u)} x_v x_w + 2x_u \sum_{v \in \Gamma (u)} x_v \leq \mu (G-u) (1-x^2) + 2x_u^2 \mu (G),
$$

implying that

$$
\frac{\mu (G-u)}{n-1} \geq \frac{\mu (G)}{n-1} \cdot \frac{1-2x_u^2}{1-x_u^2} = \frac{\mu (G)}{n-1} \cdot \left( \frac{n-2+2c}{n-1+c} \right).
$$

On the other hand, in view of $0 \leq c \leq 1$, we find that

$$
n \left( \frac{n-2+2c}{n-1+c} \right) - n + 1 - c + \frac{1}{n-1} = -1 + c + cn - c + \frac{1}{n-1} = -\frac{(1-c)^2}{n-1+c} + \frac{1}{n-1} \geq 0.
$$
Hence, inequality (1) implies that
\[
\frac{\mu(G - u)}{n - 1} \geq \frac{\mu(G)}{n - 1} \cdot \left(\frac{n - 2 + 2c}{n - 1 + c}\right) \geq \frac{\mu(G)}{n} \cdot \left(1 + \frac{c}{n - 1} - \frac{1}{(n - 1)^2}\right),
\]
completing the proof. \( \square \)

**Proof of Theorem 5**

Let \( \alpha, \beta, \gamma, K, n \), and the graph \( G \) satisfy the conditions of the theorem. We immediately see that
\[
n \geq \frac{42K + 4}{\alpha^2 \beta} > \max \left\{ \frac{15}{(1 - \beta) \alpha}; \frac{1}{\beta}; \frac{84K}{\alpha^2} \right\}.
\]

Define a sequence of graphs \( G_0, \ldots, G_k \) by the following procedure \( \mathcal{P} \):

\begin{algorithmic}
\State \text{begin}
\State \text{set} \( G_0 = G \);
\State \text{set} \( k = 0 \);
\State \text{while} \( \delta \left( G_i \right) \leq (\gamma - \alpha) \left( n - k \right) \) \text{ and } k < \lfloor \beta n \rfloor \text{ do}
\State \text{begin}
\State \text{select a unit eigenvector} \( (x_1, \ldots, x_{n-k}) \) to \( \mu \left( G_k \right) \);
\State \text{select a vertex} \( u_k \in V \left( G_k \right) \) \text{ such that} \( x_{u_k} = \min \{ x_1, \ldots, x_{n-k} \} \);
\State \text{set} \( G_{k+1} = G_k - u_k \);
\State \text{add} 1 \text{ to} \ k;
\State \text{end};
\State \text{end};
\State \text{end}.
\end{algorithmic}

Let \( H = G_k \) and note that
\[
|H| = n - k \geq n - \lfloor \beta n \rfloor \geq (1 - \beta) n.
\]

We shall show that
\[
\mu \left( H \right) > \gamma \left( 1 + \frac{4k \alpha}{7n} \right) |H|.
\]
To this end, we first prove by induction on \( i \) that
\[
\frac{\mu \left( G_i \right)}{n - i} \geq \left( 1 + \frac{3i \alpha}{5n} \right) \frac{\mu \left( G \right)}{n}
\]
for every \( i = 0, \ldots, k \).

The assertion is trivially true for \( i = 0 \). Let \( 0 \leq i \leq k - 1 \) and assume that (3) holds for \( i \); we shall prove that it also holds for \( i + 1 \). Set \( \delta = \delta \left( G_i \right), \mu = \mu \left( G_i \right) \), and note first that
\[
\delta \leq (\gamma - \alpha) \left( n - i \right),
\]
\[
\mu \geq \left( n - i \right) \left( 1 + \frac{3i \alpha}{5n} \right) \frac{\mu \left( G \right)}{n} > (n - i) \left( \gamma - \frac{K}{n^2} \right).
\]
Let \((x_1, \ldots, x_{n-i})\) be a unit eigenvector to \(\mu\), and let \(u \in V(G_i)\) satisfy \(x_u = \min \{x_1, \ldots, x_{n-i}\}\). Then Lemma 3 implies that
\[
x_u^2 \leq \frac{\delta}{\mu^2 + (n-i) \delta - \delta^2}.
\]
Noting that the right-hand side increases with \(\delta\) and decreases with \(\mu\), in view of (4) and (5), we find that
\[
x_u (n-i) \leq \frac{(\gamma - \alpha) (n-i)^2}{(n-i)^2 (\gamma - K/n^2)^2 + (n-i) (\gamma - \alpha) (n-i) - (\gamma - \alpha)^2 (n-i)^2}
\]
\[
< \frac{\gamma - \alpha}{\gamma - \alpha^2} < 1 - \frac{2\alpha}{3}.
\]
In the above derivation we used the inequalities
\[
\alpha \leq 1/4, \quad 2K/n^2 < \alpha^2, \quad 1 \geq \gamma \geq 1/2 - \alpha/4 \geq 3\alpha/4 > 2\alpha^2.
\]
Next, Lemma 4 implies that
\[
\frac{\mu (G_{i+1})}{n-i-1} \geq \frac{\mu (G_i)}{n-i} \left(1 + \frac{1}{n-i-1} \left(\frac{2\alpha}{3} - \frac{1}{n-i-1}\right)\right) \geq \frac{\mu (G_i)}{n-i} \left(1 + \frac{3\alpha}{5n}\right).
\]
Therefore,
\[
\frac{\mu (G_{i+1})}{n-i-1} \geq \left(1 + \frac{3\alpha}{5n}\right) \left(1 + \frac{3i\alpha}{5n}\right) \frac{\mu (G)}{n} \geq \left(1 + \frac{3(i+1)\alpha}{5n}\right) \frac{\mu (G)}{n},
\]
completing the induction step and the proof of (3).
Inequality (3) implies that
\[
\frac{\mu(|H|)}{|H|} = \frac{\mu (G_k)}{n-k} \geq \left(1 + \frac{3k\alpha}{5n}\right) \frac{\mu (G)}{n} \geq \left(1 + \frac{3k\alpha}{5n}\right) \left(\frac{\gamma - K}{n^2}\right)
\]
\[
= \gamma \left(1 + \frac{3k\alpha}{5n}\right) - K/n^2 \left(1 + \frac{3k\alpha}{5n}\right) > \gamma \left(1 + \frac{4k\alpha}{7n}\right) + \frac{\alpha}{42n} - \frac{2K}{n^2}
\]
\[
> \gamma \left(1 + \frac{4k\alpha}{7n}\right),
\]
as claimed.
To complete the proof of the theorem, note that, after the procedure \(\mathcal{P}\) stops, we have either \(k = \lfloor \beta n \rfloor\) or \(\delta (H) > (\gamma - \alpha) |H|\). If \(k = \lfloor \beta n \rfloor\), then
\[
\mu (H) \geq \gamma \left(1 + \frac{4 \lfloor \beta n \rfloor \alpha}{7n}\right) |H| > \gamma \left(1 + \frac{\beta \alpha}{2}\right) |H|;
\]
hence, condition (i) holds. If \( k < \lfloor \beta n \rfloor \), then \( \delta(H) > (\gamma - \alpha)|H| \), and, in view of (2), we find that

\[
\mu(H) > \gamma \left(1 + \frac{k\alpha}{2n}\right)|H| > \gamma |H|;
\]

hence, condition (ii) holds, completing the proof.

\[\square\]

**Proof of Theorem 1**

Let \( G \) be a graph of order \( n \) with \( \mu(G) > \sqrt{\frac{n^2}{4}} \). Assume first that \( \delta(G) > 2n/5 \). Since \( G \) contains a triangle, it is nonbipartite; hence, for \( n \) sufficiently large, Fact 6 implies that \( C_t \subset G \) for every \( t \leq \delta(G) + 1 \), completing the proof.

Thus, we shall assume that \( \delta(G) \leq 2n/5 \). Let

\[
\alpha = 1/10, \quad \beta = 1/2, \quad \gamma = 1/2, \quad K = 1.
\]

We have \( \delta(G) \leq (\gamma - \alpha)n \) and

\[
\mu(G) \geq \sqrt{\frac{n^2}{4}} \geq n/2 - 1/n = \gamma n - K/n.
\]

Hence, Theorem 5 implies that, for \( n \) sufficiently large, there exists an induced subgraph \( H \subset G \) with \( |H| > n/2 \), satisfying one of the following conditions:

(i) \( \mu(H) > (1/2 + 1/80)|H| \);
(ii) \( \mu(H) > |H|/2 \) and \( \delta(H) > 2|H|/5 \).

Assume first that condition (i) holds. Then, by Fact 7, we obtain

\[
k_3(H) > \left(\frac{\mu(H)}{|H|} - \frac{1}{2}\right) \frac{1}{12} |H|^3 \geq \frac{1}{80 \cdot 12} |H|^3 = \frac{1}{960} |H|^3.
\]

Thus, there is a vertex \( u \in V(H) \) contained in at least \( 3k_3(H) / |H| \geq |H|^2 / 320 \) triangles in \( H \), and so the neighborhood of \( u \) induces more than \( |H|^2 / 320 \) edges. By a theorem of Erdős and Gallai [4], the neighborhood of \( u \) contains a path \( P \) longer than

\[
\frac{2}{320} |H| \geq \frac{1}{320} n.
\]

Clearly, the path \( P \) and the vertex \( u \) form a cycle \( C_t \) for every \( t \leq n/320 \), completing the proof in this case.

If condition (ii) holds then, by \( \mu(H) > |H|/2 \), the graph \( H \) contains a triangle; thus, by Fact 6, \( C_t \subset H \) for every \( t \leq \delta(H) + 1 \), completing the proof. \[\square\]
Proof of Theorem 2

Let $G$ be a graph of order $n$ with $\mu(G) > (1/2 - \theta)n$. If $G$ is triangle-free, the proof is completed by Fact 8, so we shall assume that $G$ contains a triangle.

Assume first that $\delta(G) > 2n/5$. Since $G$ is nonbipartite, for $n$ sufficiently large, Fact 6 implies that $C_t \subseteq G$ for every $t \leq \delta(G) + 1$, completing the proof.

Thus, we shall assume that $\delta(G) \leq 2n/5$. Let

$$\alpha = 1/10 + \theta, \quad \beta = 40\theta, \quad \gamma = 1/2 - \theta, \quad K = 0.$$  

We have $\delta(G) \leq (\gamma - \alpha)n$ and $\mu(G) > 1/2 - \theta = \gamma n$. Hence, Theorem 5 implies that, for $n$ sufficiently large, there exists an induced subgraph $H \subseteq G$ with $|H| > (1 - \beta)n$, satisfying one of the following conditions:

(i) $\mu(H) > \gamma (1 + \alpha \beta/2) |H|$;
(ii) $\mu(H) > \gamma |H|$ and $\delta(H) > (\gamma - \alpha) |H|$.

Assume first that condition (i) holds. Then,

$$\mu(H) \geq \gamma (1 + \alpha \beta/2) |H| = \gamma (1/2 - \theta + (1/2 - \theta) (1/10 + \theta) 20\theta) |H|$$

$$= (1/2 - \theta - \theta + 8\theta^2 - 20\theta^3) |H| > |H|/2,$$

and so, by Theorem 1, $C_t \subseteq H \subseteq G$ for every $t < |H|/320$. This completes the proof in view of

$$|H|/320 \geq (1 - 40\theta)n/320 > n/321.$$

If condition (ii) holds then, in view of $\delta(H) > (\gamma - \alpha)|H| = 2|H|/5$, Fact 6 implies that $C_t \subseteq H$ for all $t \leq \delta(H) + 1$, unless $H$ is bipartite. To complete the proof we have to consider case of bipartite $H$. Since $H$ is triangle-free and $\mu(H) > \gamma |H| = (1/2 - \theta) |H|$, Fact 8 implies that there exists an induced bipartite subgraph $G_0 \subseteq H$ satisfying

$$|G_0| > (1 - 3\theta^{1/3}) |H| \geq (1 - 3\theta^{1/3}) (1 - \beta)n = (1 - 3\theta^{1/3}) (1 - 40\theta)n > (1 - 4\theta^{1/3})n$$

and

$$\delta(G_0) > (1 - 6\theta^{1/3}) |H| \geq (1 - 6\theta^{1/3}) (1 - \beta)n = (1 - 3\theta^{1/3}) (1 - 40\theta)n > (1 - 7\theta^{1/3})n,$$

completing the proof. \qed

Concluding remarks

It is clear that the constant 1/320 in Theorem 1 can be increased even with the present methods; thus, the following question arises:

**Question** What is the maximum $C$ such that for all positive $\varepsilon < C$ and sufficiently large $n$, every graph $G$ of order $n$ with $\mu(G) > \sqrt{[n^2/4]}$ contains a cycle of length $t$ for every $t \leq (C - \varepsilon)n$.  

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It is known ([1], p. 150) that if $G$ is a graph of order $n$ with $e(G) > \lfloor n^2/4 \rfloor$, then $G$ contains a cycle of length $t$ for every $3 \leq t \leq \lfloor n/2 \rfloor$. Thus, one can conjecture that $C = 1/2$. However, this is not true: taking the join of a complete graph of order $k = \lfloor (3 - \sqrt{5}) n/4 \rfloor$ and an empty graph of order $n - k$, we obtain a graph $H$ of order $n$ with $\mu(H) > n/2 \geq \sqrt{\lfloor n^2/4 \rfloor}$, but having no cycles longer than $2k \sim (3 - \sqrt{5}) n/2$.

Finally, a word about the project mentioned in the introduction: in this project we try to follow the following principles:

- give results that can be used as wide-range tools, like Lemmas 3 and 4, Theorem 5, and Facts 7 and 8;
- give explicit conditions for the parameters in statements, like the conditions for $\alpha, \beta, \gamma, K, n$ in Theorem 5;
- prefer simple to optimal bounds, like the factor $1/320$ in Theorem 1.

We aim to give results that can be used further, hoping to add more integrity to spectral extremal graph theory.

References


