Complete \( r \)-partite subgraphs of dense \( r \)-graphs

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Abstract

Extending a classical result of Erdős, we derive the following concise statement:

Let \( r \geq 3 \) and \( (\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3} \). Then every \( r \)-uniform graph on \( n \) vertices with at least \( \alpha n^r / r! \) edges contains a complete \( r \)-partite subgraph with \( r - 1 \) classes of size \( \left\lfloor \alpha (\ln n)^{1/(r-1)} \right\rfloor \) and one class of size \( \left\lfloor n^{1-\alpha^{r-2}} \right\rfloor \).

Our main result is a similar, but stronger statement about directed hypergraphs.

Keywords: directed hypergraph; number of edges; complete multipartite subgraph.

In this note graph means \( r \)-uniform hypergraph for some fixed \( r \geq 3 \).

Given \( c > 0 \), how large complete \( r \)-partite subgraphs must contain a graph with \( n \) vertices and \( cn^r \) edges? As shown by Erdős and Stone [1] and Erdős [2], such a graph contains a complete \( r \)-partite subgraph with each class of size \( a (\log n)^{1/(r-1)} \) for some \( a = a(c) > 0 \), independent of \( n \).

In this note we extend this fundamental result in three directions: \( c \) may be a function of \( n \), the complete \( r \)-partite subgraph may have vertex classes of variable size, and graphs may be directed.

Letting \( K_r(s_1, \ldots, s_r) \) be the complete \( r \)-partite graph with vertex classes of size \( s_1, \ldots, s_r \), we prove the following

**Theorem 1** Let \( r \geq 3 \), \( (\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3} \), and the positive integers \( s_1, \ldots, s_{r-1} \) satisfy \( s_1 s_2 \cdots s_{r-1} \leq \alpha^{r-1} \ln n \). Then every graph with \( n \) vertices and at least \( \alpha n^r / r! \) edges contains a \( K_r(s_1, \ldots, s_{r-1}, t) \) with \( t > n^{1-\alpha^{r-2}} \).

It turns out that it is easier to prove Theorem 1 in a more general setup, viz., for directed \( r \)-graphs. Thus our principal statement is the following
Theorem 2 Let \( r \geq 3, (\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3} \), and the positive integers \( s_1, \ldots, s_{r-1} \) satisfy \( s_1 s_2 \cdots s_{r-1} \leq \alpha^{r-1} \ln n \). Let \( U_1, \ldots, U_r \) be sets of size \( n \) and \( M \subseteq U_1 \times \cdots \times U_r \) satisfy \( |M| \geq \alpha n^r \). Then there exist \( V_1 \subseteq U_1, \ldots, V_r \subseteq U_r \) satisfying \( V_1 \times \cdots \times V_r \subseteq M \) and

\[
|V_1| = s_1, \ldots, |V_{r-1}| = s_{r-1}, \quad |V_r| > n^{1-\alpha^{r-2}}.
\]

We prove Theorem 2 by an involved counting argument. For a better view on the matter we give a separate theorem, hoping that it may have other applications as well.

Let \( U_1, \ldots, U_r \) be nonempty sets and \( M \subseteq U_1 \times \cdots \times U_r \). Let the positive integers \( s_1, \ldots, s_r \) satisfy \( |U_i| \geq s_i \) \((1 \leq i \leq r)\). Write \( B_M (s_1, \ldots, s_r) \) for the set of products \( V_1 \times \cdots \times V_r \subseteq M \) such that \( V_i \subseteq U_i \) and \( |V_i| = s_i \) for \( i = 1, \ldots, r \).

Theorem 3 Let \( r \geq 2 \), let \( U_1, \ldots, U_r \) be sets of size \( n \) and \( M \subseteq U_1 \times \cdots \times U_r \) satisfy \( |M| \geq \alpha n^r \). If

\[
2^r \exp \left( -\frac{1}{r} (\ln n)^{1/r} \right) \leq \alpha \leq 1
\]

and the positive integers \( s_1, s_2, \ldots, s_r \) satisfy \( s_1 s_2 \cdots s_r \leq \ln n \), then

\[
|B_M (s_1, \ldots, s_r)| \geq \left( \frac{\alpha}{2^r} \right)^{s_1 \cdots s_r} \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_r} \right).
\]

Remarks

- The relations between \( \alpha \) and \( n \) in the above theorems need some explanation. First, for fixed \( \alpha \), they show how large must be \( n \) to get valid conclusions. But, in fact, the relations are subtler, for \( \alpha \) itself may depend on \( n \), e.g., letting \( \alpha = 1/\ln \ln n \), the conclusions are meaningful for sufficiently large \( n \).

- Note that, in Theorems 1 and 2, if the conclusion holds for some \( \alpha \), it holds also for \( 0 < \alpha' < \alpha \), provided \( n \) is sufficiently large.

- As Erdős showed in [2], most graphs with \( n \) vertices and \( (1 - \varepsilon) \binom{n}{2} \) edges have no \( K_r (s, \ldots, s) \) for \( s \geq c (\log n)^{1/(r-1)} \) and sufficiently large constant \( c = c(\varepsilon) \), independent of \( n \). Hence, Theorems 1 and 2 are essentially best possible at least for fixed \( \alpha \). On the other hand, in Theorem 2, we cannot determine how large the set \( V_r \) can be, even for \( r = 3 \).

- Finally, observe that for \( r = 2 \) the relations are different, e.g., the equivalent of Theorem 2 is the following version of Lemma 2 in [3]:

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Let \((\ln n)^{-1/2} \leq \alpha < 1/2\), and let \(G\) be a bipartite 2-graph with parts of size \(n\) with at least \(\alpha n^2\) edges. Then \(G\) contains a \(K_2(s,t)\) with \(s = [\alpha^2 \ln n]\) and \(t > n^{1-\alpha}\).

Proofs

Let us start with some definitions.

Suppose that \(U_1, \ldots, U_r\) are nonempty sets and \(M \subset U_1 \times \cdots \times U_r\); let the integers \(s_1, \ldots, s_r\) satisfy \(0 < s_i \leq |U_i|\), for \(i = 1, \ldots, r\).

Define \(M' \subset U_1 \times \cdots \times U_{r-1}\) as

\[
M' = \{(u_1, \ldots, u_{r-1}) : \text{there exists } u \in U_r \text{ such that } (u_1, \ldots, u_{r-1}, u) \in M\}.
\]

For every \(R \in B_{M'}(s_1, \ldots, s_{r-1})\), let

\[
N_M(R) = \{u : u \in U_r \text{ and } (u_1, \ldots, u_{r-1}, u) \in M \text{ for every } (u_1, \ldots, u_{r-1}) \in R\},
\]

\[
d_M(R) = |N_M(R)|.
\]

For every \(v \in U_r\), let

\[
N_M(v) = \{(u_1, \ldots, u_{r-1}) : (u_1, \ldots, u_{r-1}, v) \in M\},
\]

\[
d_M(v) = |N_M(v)|,
\]

\[
D_M(v) = |\{R : R \in B_{M'}(s_1, \ldots, s_{r-1}) \text{ and } v \in N_M(R)\}|.
\]

Finally, for every integer \(s \geq 1\), let

\[
g_s(x) = \begin{cases} 
\binom{x}{s} & \text{if } x > s - 1; \\
0 & \text{if } x \leq s - 1.
\end{cases}
\]

Proof of Theorem 3 By symmetry we assume that \(s_1 \geq s_2 \geq \cdots \geq s_r\). To prove the assertion we use induction on \(r\). Let first \(r = 2\). Since \(g_{s_2}(x)\) is convex, we see that

\[
|B_M(s_1, s_2)| = \sum_{R \subset U_1, |R| = s_1} \binom{d_M(R)}{s_2} = \sum_{R \subset U_1, |R| = s_1} g_{s_2}(d_M(R)) \geq \binom{n}{s_1} g_{s_2} \left( \binom{n}{s_1}^{-1} \sum_{R \subset U_1, |R| = s_1} d_M(R) \right).
\]

On the other hand, using the convexity of \(g_{s_1}(x)\), we find that

\[
\sum_{R \subset U_1, |R| = s_1} d_M(R) = \sum_{u \in U_2} \binom{d_M(u)}{s_1} = \sum_{u \in U_2} g_{s_1}(d_M(u)) \geq n g_{s_1} \left( \frac{1}{n} \sum_{u \in U_2} d_M(u) \right) \geq n \binom{|M|/n}{s_1} \geq n \left( \frac{\alpha n}{s_1} \right).
\]
By the assumption,
\[ \alpha n \geq 4 \exp \left( \ln n - \frac{1}{2} (\ln n)^{1/2} \right) > 2 \exp \left( \frac{1}{2} \ln n \right) \geq 2 \ln n \geq 2s_1. \]

Therefore,
\[ n \left( \frac{\alpha n}{s_1} \right) \geq n \left( \frac{\alpha}{2} \right)^{s_1} \left( \frac{n}{s_1} \right), \]
and, since \( g_{s_2}(x) \) is non-decreasing, we obtain
\[ |B_M(s_1, s_2)| \geq \left( \frac{n}{s_1} \right)^{g_{s_2}} \left( n \left( \frac{n}{s_1} \right)^{-1} \left( \alpha n \right) \right) \geq \left( \frac{n}{s_1} \right)^{g_{s_2}} \left( \left( \frac{\alpha}{2} \right)^{s_1} n \right). \]

Likewise, from
\[ -\frac{1}{2} (\ln n)^{1/2} \leq \ln \frac{\alpha}{4} \leq \ln \frac{1}{4}, \]
we see that \( n \geq e^{(\ln 16)^2} \), and so,
\[ (\alpha/2)^{s_1} n \geq (\alpha/2)^{\ln n} n = n^{1+\ln \alpha/2} \geq n^{0.3} \geq 2\sqrt{\ln n} \geq 2s_2. \]

This inequality implies that
\[ |B_M(s_1, s_2)| \geq \left( \frac{n}{s_1} \right)^{g_{s_2}} \left( n \left( \frac{n}{s_1} \right)^{-1} \left( \alpha n \right) \right) \geq \left( \frac{n}{s_1} \right)^{g_{s_2}} \left( \left( \frac{\alpha}{2} \right)^{s_1} n \right), \]
completing the proof for \( r = 2 \).

Assume now the assertion true for \( r - 1 \); we shall prove it for \( r \). We first show that there exist \( W \subset U_r \) and
\[ L \subset M \cap (U_1 \times \cdots \times U_{r-1} \times W) \]
with \(|L| > (\alpha/2)n^r\) such that \( d_L(u) \geq (\alpha/2)n^{r-1} \) for all \( u \in W \). Indeed, apply the following procedure:

**Let** \( W = U_r, L = M; \)

**While** there exists an \( u \in W \) with \( d_L(u) < (\alpha/2)n^{r-1} \) **do**

**Remove** \( u \) **from** \( W \). **Remove all** \( r \)-**tuples containing** \( u \) **from** \( L \).
When this procedure stops, we have \( d_L(u) \geq (\alpha/2) n^{r-1} \) for all \( u \in W \). In addition,

\[
|M| - |L| < (\alpha/2) n^{r-1} n \leq (\alpha/2) n^r,
\]

implying that \(|L| \geq (\alpha/2) n^r\), as claimed.

Since \( g_s(x) \) is convex, we see that

\[
|B_L(s_1, \ldots, s_r)| \geq \sum_{R \in B_L(s_1, \ldots, s_r-1)} \left( \frac{d_L(R)}{s_r} \right) = \sum_{R \in B_L(s_1, \ldots, s_r-1)} g_{s_r} \left( \frac{d_L(R)}{s_r} \right)
\]

\[
\geq |B_L'(s_1, \ldots, s_{r-1})| g_{s_r} \left( \frac{\sum_{R \in B_L(s_1, \ldots, s_{r-1})} d_L(R)}{|B_L'(s_1, \ldots, s_{r-1})|} \right)
\]

\[
= |B_L'(s_1, \ldots, s_{r-1})| g_{s_r} \left( \frac{\sum_{u \in W} D_L(u)}{|B_L'(s_1, \ldots, s_{r-1})|} \right) \quad (1)
\]

On the other hand \( s_1 \cdots s_{r-1} \leq s_1 \cdots s_r \leq \ln n \). Also, for every \( u \in W \), we have

\[
\frac{d_L(u)}{n^{r-1}} \geq \frac{\alpha}{2};
\]

hence, in view of

\[
\frac{\alpha}{2} \geq 2^{r-1} e^{-\sqrt{\ln n}/r} > 2^{r-1} e^{-r' \sqrt{\ln n}/(r-1)},
\]

we can apply the induction hypothesis to the sets \( U_1, \ldots, U_{r-1} \), the numbers \( s_1, \ldots, s_{r-1} \), and the set \( N_L(u) \subset U_1 \times \cdots \times U_{r-1} \). We obtain

\[
D_L(u) \geq \left( \frac{\alpha/2}{2^{r-1}} \right)^{(r-1)s_1 \cdots s_{r-1}} \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_{r-1}} \right)
\]

for every \( u \in W \). This, together with \(|W| \geq |L|/n^{r-1} \geq \alpha n/2\), gives

\[
\sum_{u \in W} D_L(u) \geq \frac{\alpha n}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_{r-1}} \right).
\]

Note that the function \( g_{s_r}(x/k) \) is non-increasing in \( k \) for \( k \geq 1 \). Hence, from

\[
|B_L'(s_1, \ldots, s_{r-1})| \leq \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_{r-1}} \right)
\]
and (1), we obtain

\[
\left| B_L (s_1, \ldots, s_r) \right| \geq \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_{r-1}} \right) g_{s_r} \left( \left( \frac{n}{s_1} \right)^{-1} \cdots \left( \frac{n}{s_{r-1}} \right)^{-1} \sum_{u \in W} D_L (u) \right) 
\]
\[
\geq \left( \frac{n}{s_1} \right) \cdots \left( \frac{n}{s_{r-1}} \right) g_{s_r} \left( \frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} n \right). \quad (2)
\]

To continue the proof we need the following

**Claim 4** The condition

\[
2^r \exp \left( -\frac{1}{r} (\ln n)^{1/r} \right) \leq \alpha \leq 1 
\]

implies that

\[
\frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} n \geq 2s_r. \quad (4)
\]

**Proof** By a simple calculation we see that (3) implies that \( n > 16 \). Also, from (4) we have

\[
\frac{\alpha}{2} \geq e^{-\sqrt{\ln n}/r},
\]

and, by \( s_1 \cdots s_r \leq \ln n \), we obtain

\[
\frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_r} \geq \frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)\ln n} \geq e^{-\sqrt{\ln n}/r} \left( e^{-\sqrt{\ln n}/r} \right)^{(r-1)\ln n} = (e^{r-1}n)^{-\sqrt{\ln n}/r}. \quad (5)
\]

Using routine calculus, we find that the function \((2x/n)^x\) is increasing for \( x \geq 1 \) and \( n \geq 16 \). This fact, together with

\[
1 \leq s_r \leq (s_1 \cdots s_{r-1}s_r)^{1/r} \leq \sqrt[2r]{\ln n},
\]

implies that

\[
\left( \frac{2\sqrt[2r]{\ln n}}{n} \right)^{\sqrt[2r]{\ln n}} \geq \left( \frac{2s_r}{n} \right)^{s_r}. \quad (6)
\]

For \( n \geq 16 \) we easily see that

\[
\left( \frac{n}{e} \right)^{r-1} \frac{1}{2^r} \geq \left( \frac{n}{e} \right)^{2} \frac{1}{8} \geq \left( \frac{n}{8} \right)^{2} \geq \frac{n}{4} \geq \ln n,
\]

and so,

\[
(e^{r-1}n)^{-1/r} \geq \frac{2\sqrt[2r]{\ln n}}{n}.
\]
and
\[(e^{r-1}n)^{-\sqrt{\ln n}/r} \geq \left(\frac{2\sqrt{\ln n}}{n}\right)^{\sqrt{\ln n}}.\]

This, together with (5) and (6) gives
\[\frac{\alpha}{2} \left(\frac{\alpha}{2r}\right)^{(r-1)s_1\ldots s_r} \geq \left(\frac{2s_r}{n}\right)^{s_r},\]
completing the proof of the claim. \(\square\)

From (2) and the definition of \(g_{s_r}(x)\) we see that
\[|B_L(s_1, \ldots, s_r)| \geq \left(\frac{n}{s_1}\right) \cdots \left(\frac{n}{s_{r-1}}\right) \left(\frac{\alpha}{2}\right)^{(r-1)s_1\ldots s_r} \left(\frac{n}{s_1}\right) \cdots \left(\frac{n}{s_r}\right)\]
\[> \left(\frac{\alpha}{2r}\right)^{s_1\ldots s_r} \left(\frac{n}{s_1}\right) \cdots \left(\frac{n}{s_r}\right),\]
completing the induction step and the proof of Theorem 3. \(\square\)

**Proof of Theorem 2** Using the procedure in Theorem 3, we first find \(W \subset U_r\) and

\[L \subset M \cap (U_1 \times \cdots \times U_{r-1} \times W)\]
with \(|L| > (\alpha/2) n^r\) such that \(d_L(u) \geq \alpha/n^{r-1}\) for all \(u \in W\).

For every \(R \in B_{s_{r-1}}(s_1, \ldots, s_{r-1})\), the value \(d_L(R)\) is equal to the number of elements of \(L\) containing \(R\). Hence,

\[\sum_{R \in B_{s_{r-1}}(s_1, \ldots, s_{r-1})} d_L(R) = |L|\]

Likewise, for every \(u \in W\), the value \(D_L(R)\) is equal to the number of elements of \(L\) containing \(u\). Hence,

\[\sum_{u \in W} D_L(u) = |L| .\]

Let
\[t = \max \{d_L(R) : R \in B_{s_{r-1}}(s_1, \ldots, s_{r-1})\} .\]

We have
\[t^\left(\frac{n}{s_1}\right) \cdots \left(\frac{n}{s_{r-1}}\right) \geq t|B_{s_{r-1}}(s_1, \ldots, s_{r-1})| \geq |L| = \sum_{u \in W} D_L(u) .\]
(7)
To continue the proof we need the following

**Claim 5** *The condition \((\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}\) implies that*

\[
2^{r-1} \exp \left(-\frac{1}{r-1} (\ln n)^{1/(r-1)} \right) \leq \frac{\alpha}{2} \leq 1
\]

*Proof* The second inequality is obvious, so all we have to prove is that

\[
\ln \frac{\alpha}{2^r} \geq -\frac{1}{r-1} (\ln n)^{1/(r-1)}.
\]

The function \(x^x\) decreases for \(0 < x < e^{-1}\), and \(\alpha \leq r^{-3}\); hence

\[
\alpha \ln \frac{\alpha}{2^r} \geq \frac{1}{r^3} \ln \frac{1}{r^3 2^r} = -\frac{1}{r^3} (3 \ln r + r \ln 2) > -\frac{3r}{r^3} \geq -\frac{1}{r},
\]

and so,

\[
\ln \frac{\alpha}{2^r} > -\frac{1}{(r-1) \alpha} \geq -\frac{1}{r-1} (\ln n)^{-1/(r-1)},
\]

completing the proof of the claim. \(\square\)

Since for every \(u \in W\) we have

\[
d_L(u) / n^{r-1} \geq \frac{\alpha}{2},
\]

in view of Claim 5, we may apply Theorem 3 to the sets \(U_1, \ldots, U_{r-1}\), the numbers \(s_1, \ldots, s_{r-1}\), and the set \(N_L(u) \subset U_1 \times \cdots \times U_{r-1}\), thus obtaining

\[
D_L(u) \geq \left( \frac{\alpha/2}{2^{r-1}} \right)^{(r-1)s_1 \cdots s_{r-1}} \binom{n}{s_1} \cdots \binom{n}{s_{r-1}}
\]

for every \(u \in W\). This, together with \(|W| \geq |L| / n^{r-1} \geq \alpha n / 2\), gives

\[
\sum_{u \in W} D_L(u) \geq \frac{\alpha n}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} \binom{n}{s_1} \cdots \binom{n}{s_{r-1}}.
\]

Substituting this bound in (7), we find that

\[
t \geq \frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} n \geq \frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1) \alpha^{r-1} \ln n} n > \left( \frac{\alpha}{2^r} \right)^{ra^{r-1} \ln n} n.
\]

Finally, (8) gives

\[
\left( \frac{\alpha}{2^r} \right)^{ra^{r-1} \ln n} > e^{-\alpha^{r-2} \ln n} = n^{-\alpha^{r-2}},
\]

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Proof of Theorem 1 Suppose $r, \alpha, n$, and $G$ satisfy the conditions of the theorem. Let $U_1, \ldots, U_r$ be $r$ copies of the vertex set $V$ of $G$, and let $M \subset U_1 \times \cdots \times U_r$ be the set of $r$-vectors $(u_1, \ldots, u_r)$ such that $\{u_1, \ldots, u_r\}$ is an edge of $G$. Clearly, $|M| \geq r!(\alpha n^r/r!) = \alpha n^r$. Theorem 2 implies that there exists a set $V'_1 \times \cdots \times V'_r \subset M$ such that $V'_i \subset U_i$ and $|V'_i| = s_i$ for $1 \leq i < r$, and $|V'_r| > n^{1-\alpha r-2}$. Let $V_1, \ldots, V_r$ be the subsets of $V$, corresponding to $V'_1 \times \cdots \times V'_r$. The sets $V_1, \ldots, V_r$ are disjoint, for the edges of $G$ consist of distinct vertices. Hence $V_1, \ldots, V_r$ are the vertex classes of an $r$-partite subgraph of $G$ with the desired size.

References

