Cycles and paths in graphs with large minimal degree

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Abstract

Let \(G\) be a simple graph of order \(n\) and minimal degree > \(cn\) \((0 < c < 1/2)\). We prove that

1) There exist \(n_0 = n_0(c)\) and \(k = k(c)\) such that if \(n > n_0\) and \(G\) contains a cycle \(C_t\) for some \(t > 2k\), then \(G\) contains a cycle \(C_{t-2s}\) for some positive \(s < k\).

2) Let \(G\) be 2-connected and nonbipartite. For each \(\varepsilon \ (0 < \varepsilon < 1)\), there exists \(n_0 = n_0(c, \varepsilon)\) such that if \(n \geq n_0\) then \(G\) contains a cycle \(C_t\) for all \(t\) with

\[
\left\lceil \frac{2}{c} \right\rceil - 2 \leq t \leq 2(1 - \varepsilon)cn.
\]

This answers positively a question of Erdős, Faudree, Gyárfás and Schelp.

1 Introduction

Our graph-theoretic notation is standard (see [4]). In particular, all graphs are assumed to be defined on the vertex set \([n] = \{1, 2, \ldots, n\}\) and for any vertex \(i\), \(N_i\) is the set of its neighbours and \(d_i\) is its degree. We use the same name for a path and for the set of its vertices.

Recall the following result of Voss and Zuluaga [12] published in 1977.

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Theorem 1 If a 2-connected, nonbipartite graph $G$ has $n$ vertices and minimum degree $\delta$, then it contains an odd cycle of order at least $2\delta - 1$, and an even cycle of order at least $2\delta$.

In 1988 Erdős, Faudree, Gyárfás and Schelp [8] considered the problem of finding the orders of odd cycles that necessarily occur in $G$. Among other results they sketched the proof of the following assertion.

Theorem 2 Let $G$ be a 2-connected, nonbipartite graph of order $n$ and minimum degree $\geq cn$, $(0 \leq c \leq 1/3)$. For each $\varepsilon$ with $0 < \varepsilon < 1$, there exist $h(c, \varepsilon)$ and $n_0(c, \varepsilon)$, such that if $n \geq n_0(c, \varepsilon)$, then $G$ contains a cycle of order $t$ for all odd $t$ with

$$h(c, \varepsilon) \leq t \leq \frac{4}{3} (1 - \varepsilon) cn.$$

The authors of [8] further conjectured that the constant $4/3$ in the right-hand side of (1) could be replaced with $2$. The main objective of our note is the proof of this conjecture; it appears in Section 4, Theorems 6 and 9. As pointed out in [8], the constant 2 cannot be significantly improved. Indeed, fix some integer $\delta < n/2$, take the graph $K_{\delta, n-\delta}$ and add a single edge to its part of cardinality $\delta$; the resulting graph is 2-connected, nonbipartite, has minimum degree $\delta$, and contains no odd cycles of order greater than $2\delta - 1$.

In Section 2 we determine that if the minimum degree of a graph of order $n$ is $> cn$ then both the odd and even cycle orders are distributed uniformly. This result can be enhanced considerably, but we present here the least complicated version capable to support the proof of Theorem 6. In Section 3 we give several sufficient conditions on a graph for extending the order its of paths and cycles by 2.

We conclude the introduction by stating some simple bounds, related to the Zarankiewicz problem, that we shall need. The following well-known estimate is proved in [4] (p. 114, Theorem 12).

Lemma 1 Let $m, n, s, q$ be integers with $2 \leq s \leq m$ and $2 \leq q \leq n$. If $G$ is a bipartite graph with parts $A$ and $B$ ($|A| = m$, $|B| = n$) and

$$e(G) > (q - 1)^{1/s} (n - s + 1) m^{1-1/s} + (s - 1) m$$

then $G$ contains a copy of $K_{s,q}$ having $s$ vertices in $A$ and $q$ vertices in $B$. 
Corollary 1 Suppose $\alpha$ is a real number with $0 < \alpha < 1$, and $m, n, q$ are integers with $q \geq 3$, $n \geq m \geq m_0(\alpha, q)$. Let $G$ be bipartite graph with parts $A$ and $B$, $(|A| = m, |B| = n)$. If $e(G) > \alpha nm$ then $G$ contains a $K_{3,q}$ having 3 vertices in $A$ and $q$ vertices in $B$.

Proof of Corollary 1 Set $m_0(\alpha, q) = 4q\alpha - 3$. Assuming the opposite, from Lemma 1 with $s = 3$, we have

$$e(G) \leq (q - 1)^{1/3} (n - 2) m^{2/3} + 2m.$$  

From $e(G) > \alpha nm$ we obtain $\alpha nm < q^{1/3} nm^{2/3} + 2m$ and therefore,

$$\alpha m^{1/3} < q^{1/3} + \frac{2m^{1/3}}{n} < q^{1/3} + \frac{2}{q^{2/3}} \leq \frac{5}{3} q^{1/3} \leq 4^{1/3} q^{1/3}.$$  

Hence, $m < 4q\alpha^{-3}$, a contradiction. □

Letting $q = 3$, we obtain immediately the following.

Corollary 2 Suppose $\alpha$ is a real number with $0 < \alpha < 1$ and $t$ is an integer. There is an integer $t_0 = t_0(\alpha)$ such that if $t \geq t_0$ and $G$ is bipartite graph with both parts of cardinality not exceeding $t$, and with $e(G) > \alpha t^2$, then $G$ contains a $K_{3,3}$.

2 Distribution of path and cycle orders

In this section we prove that if the minimum degree of a graph $G$ is relatively large, then the gaps in the orders of cycles occurring in $G$ are bounded. More precisely, the following theorem holds.

Theorem 3 Suppose $c$ is a real number with $0 < c < 1$. There exist $n_0 = n_0(c)$ and $k = k(c)$ such that for every graph $G = G(n)$ with $n > n_0$ and minimum degree $\delta > cn$, if $C_t \subset G$ for some $t > 2k$, then $C_{t-2s} \subset G$ for some positive $s < k$.

Observe that when $G$ is Hamiltonian and $n$ is large, the last theorem implies that $G$ contains cycles $C_{n-2s_1}, C_{n-2s_2}, \ldots, C_{n-2s_i}, \ldots, C_{n-2s_p}$, such that $s_1 < k$, and $s_{i-1} < s_i < s_{i-1} + k$, for $i = 2, \ldots, p - 1$, and $n - 2s_p \leq 2k$. Theorem 3 can be proved directly, but to avoid a repetition, we shall prove a more technical result that will be used later in the proof of Theorem 6.
Theorem 4 Suppose $c$ is a real number with $0 < c < 1$ and $m \geq 2$ is an integer. There exist $n_0 = n_0 (c, m)$ and $k_0 = k_0 (c)$, such that for every graph $G = G (n)$ with $n > n_0$, and minimum degree $\geq cn$, if $M \subset V (G)$ is a set of cardinality $m$, and $P$ is a $p$-path ($p \geq 2k_0$) joining two distinct vertices $u, v \in M$ with $P \cap M = \{u, v\}$, then there is a $(p - 2s)$-path $Q$ ($0 < s < k_0$) joining $u$ to $v$ and with $Q \cap M = \{u, v\}$.

**Proof of Theorem 4** Set

$$k_0 (c) = \max \left( t_0 \left( \frac{c}{8} \right), \frac{48}{c} \right)$$

($t_0 (x)$ is the function from Corollary 2), and let $P$ be a path on $p \geq 2k_0$ vertices joining $u$ to $v$ ($u, v \in M$), and such that $P \cap M = \{u, v\}$.

**Sketch of the proof.** The proof involves consideration of two cases. If the path $P$ has many chords then we can drop out an even number $< 2k_0$ of its vertices and reorder the rest of them obtaining the required shorter path $Q$. If $P$ has few chords, since the minimum degree is large, then there are many edges joining vertices from $P$ to vertices from $V (G) \setminus (P \cup M)$. We find 6 vertices from $P$ which are close to each other and all of them are adjacent to a single vertex $x \in V (G) \setminus (P \cup M)$. Then, including the vertex $x$, dropping a bounded number of vertices of $P$ and reordering the rest of them, we obtain the required path $Q$.

**Details of the proof.** Assume, wlog, $P = u, 1, ..., p - 2, v$; let $A$ be the sequence 1, ..., $p - 2$. Clearly, $A \cap M = \emptyset$. Partition the sequence $A$ into contiguous disjoint intervals of length $k$ so that only the last interval is possibly shorter than $k_0$; denote those intervals by $I_1, I_2, ..., I_s$, where $s = [(p - 2) / k_0]$.

Suppose there exist two intervals $I_i$ and $I_j$ ($i < j$), with

$$e (I_i, I_j) > \frac{c}{8} k^2.$$

By our selection, $k_0 \geq t_0 (c/8)$, and thus, from Corollary 2, there exists a copy of $K_{3,3}$ with one part in $I_i$ and one part in $I_j$. Suppose $a_1 < a_2 < a_3$ are its vertices in $I_i$ and $b_1 < b_2 < b_3$ are its vertices from $I_j$. Naturally, all $a_k$s are connected to all $b_k$s and $a_3 < b_1$, as $i < j$. Consider the two paths

$$P_1 = u, 1, ..., a_1, b_1, b_1 - 1, ..., a_2 + 1, a_2, b_3, b_3 + 1, ..., p - 2, v$$

and

$$P_2 = u, 1, ..., a_2, b_1, b_1 - 1, ..., a_3 + 1, a_3, b_3, b_3 + 1, ..., p - 2, v.$$
We see that the order of $P_1$ is $p - (b_3 - b_1 - 1) - (a_2 - a_1 - 1)$ and the order of $P_2$ is $p - (b_3 - b_1 - 1) - (a_3 - a_2 - 1)$. Furthermore, we have

$$0 < b_3 - b_1 + a_2 - a_1 - 2 < 2k_0$$
$$0 < b_3 - b_1 + a_3 - a_2 - 2 < 2k_0,$$

so if either $(b_3 - b_1 + a_2 - a_1)$ or $(b_3 - b_1 + a_3 - a_2)$ is even, the proof will be completed. Assuming they both are odd we immediately obtain that $a_3 - a_1$ is even. Exactly in the same way we obtain that $b_3 - b_1$ is even. Now consider the path $P_3 = u, 1, ... , a_1, b_1, b_1 - 1, ... , a_3 + 1, a_3, b_3, b_3 + 1, ... , p - 2, v.$

The order of this path is $p - (b_3 - b_1 - 1) - (a_3 - a_1 - 1)$, so it has the same parity as $P$ and

$$0 < b_3 - b_1 - 1 + a_3 - a_1 - 1 < 2k_0.$$

This completes the proof in case (3) holds. Therefore, we may and shall assume

$$e(I_i, I_j) \leq \frac{c}{8}k^2$$

for every $i, j (1 \leq i < j \leq s)$.

Suppose next there is an interval $I_j$ such that

$$\sum_{i \in I_j} |N_i \setminus A| > 5n + mk_0.$$  

(5)

Then, by the pigeonhole principle, there is a vertex $x \in V \setminus (A \cup M)$ that is adjacent to at least 6 distinct vertices from $I_j$. Suppose $x$ is adjacent to $a_1, a_2, a_3, a_4, a_5, a_6 \in I_j$ ($a_1 < a_2 < a_3 < a_4 < a_5 < a_6$). Consider the paths

$$P_1 = u, ..., a_1, x, a_4, ..., v$$
$$P_2 = u, ..., a_1, x, a_5, ..., v$$
$$P_3 = u, ..., a_1, x, a_6, ..., v$$

Clearly, their orders are $p - (a_4 - a_1 - 2), p - (a_5 - a_1 - 2),$ and $p - (a_6 - a_1 - 2)$, respectively. It is obvious that

$$0 < a_4 - a_1 - 2 < 2k_0,$$
$$0 < a_5 - a_1 - 2 < 2k_0,$$
$$0 < a_6 - a_1 - 2 < 2k_0.$$
so if either \((a_4 - a_1)\), or \((a_5 - a_1)\), or \((a_6 - a_1)\) is even, the proof will be completed. If they all are odd then both \(a_6 - a_5\) and \(a_5 - a_4\) are even, and consequently greater than 1. Then the path

\[ Q = u, \ldots, a_4, x, a_6, \ldots, v \]

has the required properties; the proof is completed if (5) holds.

Assume hereafter (5) does not hold for any \(j\). Thus, if \(n > mk_0\) then we have

\[
\sum_{i \in I_j} |N_i \backslash A| \leq 5n + mk_0 \leq 6n \tag{6}
\]

for every \(I_j\). Furthermore, count all edges which have an end in \(A\). We immediately see

\[
\sum_{i \in A} d_i \geq (p - 2) \delta \geq (p - 2) cn. \tag{7}
\]

On the other hand, we obviously have,

\[
\sum_{i=1}^{p-2} d_i = \sum_{j=1}^{s} \sum_{i \in I_j} d_i = \sum_{i=1}^{s} 2e(I_i) + \sum_{1 \leq i < j \leq s} 2e(I_i, I_j) + \sum_{j=1}^{s} \sum_{i \in I_j} |N_i \backslash A| \\
\leq 2s \left( \frac{k}{2} \right) + \sum_{1 \leq i < j \leq s} 2e(I_i, I_j) + \sum_{j=1}^{s} \sum_{i \in I_j} |N_i \backslash A|.
\]

This, in view of (4) and (6), yields for \(n > mk_0\)

\[
\sum_{i=1}^{p-2} d_i \leq 2s \left( \frac{k}{2} \right) + \frac{c}{2} k_0 \left( \frac{s}{2} \right) + 5ns \\
< sk_0^2 + \frac{1}{8} ck_0^2 s^2 + 6ns.
\]

From \(s = \lceil (p - 2)/k \rceil\) we have \(s \leq 2(p - 2)/k_0\). Therefore, estimating \(s\) from above,

\[
sk_0^2 + \frac{1}{8} ck_0^2 s^2 + 6ns < 2k_0 (p - 2) + \frac{1}{2} c (p - 2)^2 + \frac{10n (p - 2)}{k_0}.
\]

From (7), we obtain

\[
2k + \frac{1}{2} c (p - 2) + \frac{12n}{k} \geq cn
\]
and therefore, since \( n \geq p - 2, \)
\[
24n + 4k_0^2 > k_0cn.
\]
From (2) we have \( kc \geq 48 \) and hence, a contradiction for large \( n. \square \)

Theorem 3 follows easily from Theorem 4; select two consecutive vertices \( u \) and \( v \) of a cycle \( C_p = u, 1, ..., p - 2, v, u \); let \( M = \{u, v\} \), and replace the path \( u, 1, ..., p - 2, v \) by some path of order \( p - 2s \), where \( s \) is positive and is bounded by a function depending only on \( c \).

## 3 Path Extensions

In this section we prove three technical results that support the proof of Theorem 6. We hope, however, that they might be applied to other related problems.

Recall the following well-known result of Erdős and Gallai [9] relating the size of a graph and the order of its maximal path.

**Theorem 5** Suppose \( k \geq 1 \) is an integer. Every graph on \( n \) vertices and more than \( kn \) edges contains a \( (2k + 2) \)-path.

It is easy to derive the following corollary.

**Corollary 3** Suppose \( k \geq 1 \) is an integer and \( G \) is a bipartite graph with parts \( A \) and \( B \). If \( G \) has more than \( k(\vert A \vert + \vert B \vert) \) edges, then for every \( j \in [k] \), it contains a \( (2j + 1) \)-path with both endvertices in \( A \).

Next, we shall give a sufficient condition for extending the order of a path.

**Lemma 2** Suppose \( k \) and \( m \) are integers \( (2 \leq k < m) \), and \( G \) is a graph. Let \( P \) be a \( p \)-path in \( G \) and \( M \subset V(G) \setminus P \) be a set of cardinality \( m \). If
\[
e(M, P) > \frac{(p + 1)m + (k - 1)(m + p - 1)}{2},
\]
then for every \( s \in [k] \), the graph \( G \) contains a \( (p + s) \)-path \( Q \) having the same endvertices as \( P \), and with \( P \subset Q \subset P \cup M \).
Proof of Lemma 2 Let $P = 1, ..., p$ and let $T$ denote the edge set

$$(1, 2), (2, 3), ..., (p - 1, p).$$

It is not difficult to prove that for every vertex $x \in V(G) \setminus P$, the number of triangles containing $x$ and an edge from $T$ is at least

$$2 |N_x \cap P| - p - 1. \quad (8)$$

Denote by $t(M, P)$ the number of all triangles in $G$ having an edge from $T$ and a vertex from $M$. Summing (8) for all $x \in M$, we see

$$t(M, P) \geq \sum_{x \in M} 2 |N_x \cap P| - (p + 1) = 2e(M, P) - (p + 1) m > (k - 1) (m + p - 1).$$

Next, consider an auxiliary bipartite graph $H$ with parts $M$ and $T$; a vertex $u \in M$ is joined to a vertex $(v, v + 1) \in T$ iff there is a triangle in $G$ containing both $u$ and $(v, v + 1)$. Clearly, $H$ has $m + p - 1$ vertices and exactly $t(M, P)$ edges. From Theorem 5, since $H$ contains more than $(k - 1) (m + p - 1)$ edges, $H$ contains a path on $2k$ vertices, and consequently, a $k$-matching. Thus, there are $k$ distinct edges

$$(v_1, v_1 + 1), ..., (v_k, v_k + 1)$$

from $T$ and $k$ distinct vertices $x_1, ..., x_k$ from $M$, such that, for every $i \in [k]$, $x_i$ is joined to $v_i$ and to $v_i + 1$. Fix $s \in [k]$, and for every $i = [s]$ replace the edge $(v_i, v_i + 1)$ by the path $(v_i, x_i, v_i + 1)$. It is easy to verify that the resulting path $Q$ has the required properties. □

The following is a simplified version of a lemma that has been proved in [10].

Lemma 3 Suppose $p \geq 2$ and $k \geq 1$ are integers. Let $G$ be a graph of order $n$, $P = u, ..., v$ be a $p$-path in $G$, and $M \subset (N_u \cap N_v) \setminus P$ be of cardinality $m$. If either

$$e(M) > km \quad (9)$$

or

$$e(M, V(G) \setminus (M \cup P)) > k (n - p) \quad (10)$$

holds, then $G$ contains a cycle $C_{p + 2j + 1}$ for every $j = 0, ..., k$. 

Proof of Lemma 3 Let $P = 1, 2, ..., p$ be a path and $M \subset N_1 \cap N_p \cap (V(G) \setminus P)$. Clearly, both (9) and (10) imply $M \neq \emptyset$. Since for every vertex $v \in M$ the sequence $1, 2, ..., p, v, 1$ is an $(p + 1)$-cycle in $G$, the assertion is proved for $k = 0$. If (9) holds, then, from Theorem 5, $G[M]$ contains a $(2k + 2)$-path $Q$. Since all vertices of $Q$ are adjacent to 1 and $p$, and are distinct from any vertex of $P$, we can assemble a copy of $C_{p+2j+1}$ for every $j \in [k]$. Thus, the assertion is proved under the stipulation (9).

Assume (10) holds. Let $B = V(G) \setminus (P \cup M)$. Clearly, $|B| = n - p - m$. Consider the bipartite graph induced by the sets $M$ and $B$ and fix $j = [k]$. From Corollary 3, there exists a $(2j + 1)$-path $Q$ with both ends in $M$ and having no vertices in common with $P$. Since both 1 and $p$ are joined to every vertex of $M$, $Q$ together with $P$ is a cycle $C_{p+2j+1}$. □

Lemma 2 and Lemma 3 can be combined to give the following corollary.

Corollary 4 Suppose $p \geq 2$, $r \geq 2$, $m > 2r$ are integers. Let $G$ be a graph of order $n$, $P = (u, ..., v)$ be a $p$-path in $G$. If $M \subset V(G) \setminus P$ is a set of common neighbors of $u$ and $v$ with $|M| = m$ and

$$\sum_{i \in M} d_i \geq 3rm + rn + \frac{mp}{2},$$

then $G$ contains a cycle $C_{p+2j+1}$ for every $j = 0, ..., r$.

Proof of Corollary 4 We obviously have

$$\sum_{i \in M} d_i = 2e(M) + e(M, V(G) \setminus (M \cup P)) + e(M, P).$$

From Lemma 2, if

$$e(M, P) > \frac{(p + 1)m + (2r - 1)(m + p - 1)}{2}$$

then for every $j \in [2r]$, the graph $G$ contains a $(p + j)$-path $Q$ with the same endvertices as $P$ and $P \subset Q \subset P \cup M$. Since $j \leq 2r < m$, we see $M \setminus Q \neq \emptyset$. Fix some $v \in M \setminus Q$. Note that $v$ is adjacent to both endvertices of $Q$, thus, we have a cycle $C_{p+j+1}$ for all $j \in [2r]$.

From Lemma 3, if for some $s \in [r]$ there is no $C_{p+2s+1}$, then neither (9) nor
(10) hold. Hence, we obtain
\[
\sum_{i \in M} d_i \leq 2e(M) + e(M, V(G) \setminus (M \cup P)) + e(M, P)
\leq 2rm + r(n - p) + \frac{(p + 1)m + (2r - 1)(m + p - 1)}{2}
= \frac{4rm + 2r(n - p) + (p + 1)m + (2r - 1)(m + p - 1)}{2}
< 3rm + rn + \frac{mp}{2}.
\]
\[\square\]

4 Domains of pancyclicity

Recall that a graph is called weakly pancyclic if it contains all cycles between its girth and circumference (see [2], [7], [5], and [6].)

Suppose \( G \) is a graph with girth \( g(G) \) and circumference \( c(G) \). An interval \([a, b]\) of integers with \( g(G) \leq a \leq b \leq c(G) \) is said to be a domain of pancyclicity of \( G \) if for every \( t \) with \( a \leq t \leq b \) there is a cycle \( C_t \in G \).

It is of interest to find sufficient conditions for the existence of large domains of pancyclicity in a graph \( G \). The next theorem gives a result in that direction.

**Theorem 6** Let \( G \) be a 2-connected, nonbipartite graph of order \( n \) and \( \delta(G) > cn, \ (0 < c < 1/2) \). For each \( \varepsilon > 0, \ (1 > \varepsilon > 0) \), there exist \( n_0(c, \varepsilon) \) and \( t_0(c) \) such that if \( n \geq n_0(c, \varepsilon) \) then \( G \) contains a cycle \( C_t \) for every \( t \) with
\[
t_0(c) < t \leq 2 (1 - \varepsilon) cn.
\]

**Proof of Theorem 6** We do not give an explicit description of \( n_0(c, \varepsilon) \); such a description could be easily derived from our arguments. Let \( t_0(c) = 2k_0(c) \), where \( k_0(c) \) is the function from Theorem 4. Set \( k_0 = k_0(c) \), and suppose \( G \) does not contain a \( C_t \), where
\[
2k_0 < t < (2 - 2\varepsilon) cn.
\]

**Sketch of the proof.** First, we find a cycle \( C \) of order \( g \) with
\[
(2 - 2\varepsilon) cn < g < (2 - \varepsilon) cn
\]
and $g$ of the same parity as $t$. Next, we show that the graph induced by $C$ cannot have large size, since, from a theorem of Bondy, it will contain cycles of all orders up to $g$ including $C_t$. Hence, since the minimum degree of $G$ is large, the number of edges joining vertices from $C$ to vertices outside $C$ is at least $\varepsilon c ng/2$. Then we find two vertices $u, v \in C$ at even and bounded distance along $C$, and having as many neighbors in common outside $C$ as we might need. We apply Theorem 4 to the long path joining $u$ to $v$ along $C$, and obtain a path $P$ joining $u$ to $v$, and of order less then $t$ by a constant. Using Corollary 4 from the previous section, we complete $P$ to a cycle of order $t$.

**Details of the proof.** From Theorem 1, $G$ contains odd and even cycles of order at least $2\lceil cn \rceil - 1$. From Theorem 3 there exists $n_0 = n_0(c)$, such that for $n > n_0$, if $C_h \subset G$ ($h > 2k_0$), then $C_{h-2s} \subset G$ for some positive $s < k_0$. Select a cycle $C_h \subset G$ for some $h \geq 2\lceil cn \rceil - 1$ with the same parity as $t$. Applying repeatedly Theorem 3 we see that $G$ contains cycles $C_{h-2s_1}, C_{h-2s_2}, ..., C_{h-2s_p}$ such that $s_1 < k_0$, $s_{i-1} < s_i < s_{i-1} + k_0$ for $i = 2, ..., p-1$, and $h - 2s_p \leq 2k_0$. Let $n > \max(n_0, 2k_0/\varepsilon c)$. Clearly, at least one of the cycles constructed above must have order $g$ with

$$
(2 - 2\varepsilon) cn < g < (2 - \varepsilon) cn
$$

and $g$ of the same parity as $t$. Fix such a cycle $C$, say $C = 1, 2, ..., g, 1$, and consider the graph $G[C]$ induced by $C$. If

$$
e (G[C]) > \left\lfloor \frac{g^2}{4} \right\rfloor
$$

then, from a theorem of Bondy [3], the graph $G[C]$ is pancyclic and thus, $C_t \subset G$, a contradiction; thus, assume (14) does not hold. We obtain

$$
\sum_{i \in C} |N_i \setminus C| > g \delta - 2 \left\lfloor \frac{g^2}{4} \right\rfloor \geq g \delta - \frac{g^2}{2}
$$

$$
\geq g \left( cn - \left(1 - \frac{\varepsilon}{2}\right) cn \right) = \frac{\varepsilon c}{2} gn.
$$

By averaging, we see that for every $m$ ($0 < m \leq n$) there are $m$ consecutive vertices $i_1, ..., i_m$ along $C$ with

$$
\sum_{j=1}^{m} |N_{i_j} \setminus C| > \frac{\varepsilon c}{2} nm.
$$

(15)
Set

\[ q = \left\lceil \frac{2k_0}{\varepsilon c} \right\rceil, \]
\[ m = m_0 \left( \frac{\varepsilon c}{2}, q \right), \]

where \( m_0 (x, y) \) is the function from Corollary 1, and let

\[ n > \max \left( \frac{m}{2 (1 - \varepsilon)c}, \frac{3m}{\varepsilon c} \right). \tag{16} \]

From (13), \( g > 2(1 - \varepsilon)cn \), and thus \( g > m \). According to (15), there are \( m \) consecutive vertices \( i_1, \ldots, i_m \) from \( C \) with

\[ \sum_{j=1}^{m} |N_{i_j} \setminus C| > \frac{\varepsilon c}{2}nm. \tag{17} \]

Assume, wlog, that these are exactly the vertices \( 1, \ldots, m \). Set \( A = [m] \), \( B = V(G) \setminus C \), and consider the bipartite graph \( G(A, B) \) with parts \( A \) and \( B \). Clearly, from (13),

\[ |B| = |V(G) \setminus C| \geq n - (2 - \varepsilon)cn > \varepsilon cn, \]

and thus, as \( n \geq 3m/\varepsilon c \), we have \( |B| \geq 3m > q \). From (17), we see \( e(A, B) > \varepsilon c nm/2 \). Apply Corollary 1 to the graph \( G(A, B) \) with parts \( A \) and \( B \), and with \( \alpha = \varepsilon c/2 \). We have \( |B| \geq |A| = m \), and \( e(A, B) > \varepsilon c nm/2 > \alpha m |B| \). Since \( m \geq m_0 (\alpha, q) \), \( G(A, B) \) contains a copy of \( K_{3,q} \) having 3 vertices in \( A \) and \( q \) vertices in \( B \).

Fix a copy of \( K_{3,q} \); let \( u < v < w \) be its vertices from \( A \), and \( M_0 \) be the set of its vertices in \( B \). It is easy to see that one of the numbers \( w - v \), \( w - u \) and \( v - u \) is even. Assume, wlog, \( v - u \) is even. Let \( Q \) be the path joining \( v \) to \( u \) along \( C \), more precisely \( Q = v, v + 1, \ldots, g, 1, \ldots u \). We see that the order of \( Q \) is \( g - 2r + 1 \), where \( 2r = v - u \).

Consider the family \( \mathfrak{P} \) of all paths joining \( u \) to \( v \), that have no vertices in common with \( M_0 \), and whose order is of the same parity as \( t + 1 \). Clearly, \( Q \in \mathfrak{P} \). From Theorem 4, with \( M = M_0 \cup \{u, v\} \), for every \( P \in \mathfrak{P} \) with \( |P| > 2k_0 \), there is some \( P' \in \mathfrak{P} \) of order \( |P'| = |P| - 2j \), where \( j < k_0 \) is a positive integer. Applying Theorem 4 repeatedly we find a path \( P \in \mathfrak{P} \), of order \( p \), such that \( p < t < p + 2k_0 \). We immediately have

\[ p \leq t - 1 < 2(1 - \varepsilon)cn. \]
Apply Corollary 4 to the graph $G$, the path $P$, the set $M = M_0 \cup \{u, v\}$, and with $r = k_0$. Since $G$ does not contain $C_t$, we have

$$3k_0m + k_0n + \frac{mp}{2} \geq \sum_{i \in M_0} d_i \geq \delta m > cmn,$$

and since $p < 2(1 - \varepsilon)cn$, we easily see $3k_0m + k_0n \geq \varepsilon cmn$. By our selection, $m > q \geq 2k_0/\varepsilon c$, and hence $\varepsilon cm > 2k_0$, yielding $3k_0m + k_0n \geq 2k_0n$. Therefore, $n \leq 3m$, a contradiction with (16).

It is relatively easy to find the exact lower bound in (11). This, however, requires a different approach. We start by stating an assertion proved by Häggkvist in [11] and independently in [8].

**Theorem 7** For every integer $k \geq 2$ there exists an $n_0 = n_0(k)$, such that if $G$ is a nonbipartite, 2-connected graph on $n > n_0$ vertices with

$$\delta(G) > \frac{2n}{2k+1},$$

then $G$ contains a $C_{2k-1}$.

On the other hand, the following result has been proved in [10].

**Theorem 8** For every two positive integers $p$ and $m$ there exist $n_0 = n_0(p, m)$ and $c = c(p, m)$, such that for every nonbipartite graph $G$ on $n > n_0$ vertices and

$$\delta(G) \geq \frac{n}{2(2p+1)} + c,$$

if $C_{2s+1} \subset G$, for some $p \leq s \leq 4p + 1$, then $C_{2s+2j+1} \subset G$ for every $j \in [m]$.

Combining the last two theorems we establish the following corollary.

**Corollary 5** For every two integers $k, m$ ($k \geq 2$, $m \geq 1$) there exist $n_0 = n_0(k, m)$ and $c = c(k, m)$, such that if $G$ is a nonbipartite, 2-connected graph on $n > n_0$ vertices and

$$\delta(G) > \frac{2n}{2k+1},$$

then $G$ contains a $C_j$ for $\max(2k-2, 3) \leq j \leq 2k+2m$. 
Proof of Corollary 5 From Theorem 7 $G$ contains a $C_{2k-1}$. On the other hand,
\[ \frac{2}{2k+1} = \frac{1}{2(k/2 + 1/4)} = \frac{1}{2 (\lceil (k + 2)/4 \rceil + 1)} + \varepsilon_k \]
for some $\varepsilon_k > 0$. Setting $p = \lceil (k + 2)/4 \rceil$ we easily obtain $p \leq k - 1 \leq 4p$.
Since
\[ \delta > \left( \frac{1}{2(2p+1)} + \varepsilon_k \right) n, \]
from Theorem 8 with $s = k-1$, for $n > n_0 (p, m)$, $G$ contains a cycle $C_{2k+2j−1}$ for every $j = 0, 1, ..., m$. Therefore, we proved the existence of all odd cycles within the required range.

On the other hand, if a graph of order $n$ has more than $hn^{2−1/h}$ edges, then, for $n$ sufficiently large, it contains a $K_{h,h}$ (see [4], p. 114), and consequently, all even cycles up to $C_{2h}$. For $h = k + m$, and $n$ large enough, this implies the existence of all even cycles in the required range. □

Theorem 6 and Corollary 5 immediately yield the following theorem.

Theorem 9 Let $G$ be a 2-connected, nonbipartite graph of order $n$ and $\delta (G) > cn$, $(0 < c < 1/2)$. For each $\varepsilon > 0$, $(1 > \varepsilon > 0)$, there exists $n_0 (c, \varepsilon)$, such that if $n \geq n_0 (c, \varepsilon)$ then $G$ contains a cycle $C_t$ for every $t$ with
\[ \left\lceil \frac{2}{c} \right\rceil - 2 \leq t \leq 2 (1 - \varepsilon) cn. \]

Note that the "blown-up" $C_{2k+1}$ on $n$ vertices shows that the lower bound in the above theorem is tight; still in some cases we can decrease it by 1 using a more complicated expression.

We believe the following is true, although our methods are not applicable.

Conjecture 1 Let $G$ be a 2-connected, nonbipartite graph of order $n$ and minimum degree $\delta > cn$, $(0 < c < 1/2)$. There exists an $n_0 = n_0 (c)$, such that for $n \geq n_0$, $G$ contains a cycle $C_t$ for all $t$ with
\[ \left\lceil \frac{2}{c} \right\rceil - 2 \leq t \leq 2 \delta. \]

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