Cycle lengths in graphs with large minimum degree

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Abstract

Our main result is the following theorem.

Let \( k \geq 2 \) be an integer, \( G \) be a graph of sufficiently large order \( n \), and \( \delta(G) \geq n/k \). Then:

(i) \( G \) contains a cycle of length \( t \) for every even integer \( t \in [4, \delta(G) + 1] \).

(ii) If \( G \) is nonbipartite then \( G \) contains a cycle of length \( t \) for every odd integer \( t \in [2k-1, \delta(G) + 1] \), unless \( k \geq 6 \) and \( G \) belongs to a known exceptional class.

1 Introduction

Our graph theoretic notation is standard (e.g., see [2]); we write \( ec(G) \) and \( oc(G) \) for the lengths of the longest even and odd cycles in a graph \( G \).

In this paper we prove that if \( c > 0 \), every nonbipartite graph \( G \) of sufficiently large order \( n \) and with minimum degree \( \delta(G) \geq cn \) contains almost \( \delta(G) \) cycles of consecutive lengths up to \( \delta(G) + 1 \), unless \( G \) belongs to a known exceptional family.

The existence of arithmetic progressions of cycles lengths in graphs with sufficiently large minimum degree has been intensively studied (e.g., see [5], [6], [8], [10], [11], and [13]). Such questions happen to be more intricate for odd than for even cycles. H"{a}ggkvist [13] constructed a nonbipartite graph \( G = G(n) \) with \( \delta(G) = n/6 \) and no odd cycles longer than 3. His example is

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constructed as follows: take three vertex-disjoint copies of $K_{3,3}$, select a vertex in each of them and join every two of the selected vertices. This construction was recently generalized in [12].

For every integer $k \geq 6$, let $\mathcal{B}_k(n)$ be the family of all nonbipartite graphs $G$ of order $n$ with $\delta(G) \geq n/k$ and no odd cycles longer than $k/2$. The families $\mathcal{B}_k(n)$ have been introduced and investigated in [12]; we summarize their most important properties in Section 2.1.

Our main result in this paper is the following theorem.

**Theorem 1** Let $k \geq 2$ be an integer, $G$ be a graph of sufficiently large order $n$, and $\delta(G) \geq n/k$. Then,

(i) $C_t \subseteq G$ for every even integer $t \in [4, \delta(G) + 1]$.

(ii) If $G$ is nonbipartite then $C_t \subseteq G$ for every odd integer $t \in [2k - 1, \delta(G) + 1]$, unless $k \geq 6$ and $G \in \mathcal{B}_k(n)$.

This result is best possible. Indeed, suppose $G$ is a union of $s$ vertex-disjoint cliques of equal order. Then $\delta(G) = n/s - 1$ and the set of cycle lengths of $G$ is $[3, \delta(G) + 1]$, so the upper bounds in (i) and (ii) cannot be increased. A blown-up cycle of order $2k - 1$ shows that the lower bound in (ii) cannot be decreased.

Our proof of Theorem 1 summarizes a number of preliminary results, some of which are of independent interest. Our most important tool is Theorem 11 - a technical version of a theorem of Gould, Haxell, and Scott, stated in Section 2.4.

To give the reader a better view of our paper we shall outline briefly the main ingredients of the proof of Theorem 1.

The existence of even cycles whose lengths are in the lower range of the interval $[4, \delta(G)]$ is proved by Bondy and Simonovits [5]. The corresponding result for odd cycles is given in Lemma 18 and follows from results of Häggkvist [13] and of Győri, Nikiforov, and Schelp [12]. The existence of both odd and even cycles of length up to $\delta(G) - K$ for constant $K$ follows by the aforementioned theorem of Gould, Haxell, and Scott. Thus, the bulk of our proof will be dedicated to the existence of cycles of length close to $\delta(G)$. This innocent looking issue demands rather involved arguments that, we believe, cannot be replaced by simple refinement of previous techniques. Our main idea is as follows: if $\delta(G) < |G|/2 + 1$, find a cycle of the desired parity of length much greater than $\delta(G)$ and apply Theorem 11; if $\delta(G) \geq |G|/2 + 1$, complete the proof using Proposition 6. Most of the machinery needed appears in Lemmas 9 and 14, and Theorem 15.
2 Main Results

This section is organized as follows: We first briefly discuss the family $B_k(n)$. Next in 2.2 we introduce the notion of tight graphs and formulate some results involving path and cycle lengths. In 2.3 and 2.4 we state and prove several results used in the proof of Theorem 1; the proof itself is presented in 2.5.

2.1 The family $B_k(n)$

The structure of the graphs of $B_k(n)$ was essentially determined in [12] where the following theorem is proved.

**Theorem 2** Let $k \geq 1$, $n \geq 10(2k + 1)^2$ be integers and $G$ be a graph of order $n$ with $\delta(G) \geq \frac{n}{4k+2}$ and no odd cycles of order greater than $2k + 1$. Then there is a set $V_0$ of at most $2k$ vertices and a set $E_0$ of at most $k^2$ edges in $G$ such that $G - V_0$ and $G - E_0$ are bipartite graphs.

In the proof of Theorem 2, for every graph $G \in B_k(n)$, it is shown that:
(i) $G$ is connected but not 2-connected;
(ii) the endblocks of $G$ are bipartite graphs of order at least $2\delta(G)$.

The following example describes a typical $G \in B_k(n)$, for $k = 4l + 2$.

**Example 3** Fix two positive integers $k$ and $\delta$, $(\delta \geq 5)$; select $2l + 1$ vertex-disjoint complete bipartite graphs $K_{\delta, \delta}$; select a vertex from each of them and join each pair of the selected vertices.

The obtained graph has $2\delta (2l + 1) = k\delta$ vertices, its minimum degree is $\delta$, and it has no odd cycles longer than $2l + 1$.

2.2 Tight and loose graphs

A graph is called *tight* if $\delta(G) \geq \lfloor |G| / 2 \rfloor + 1$, and *loose* otherwise. Implicit in [9] (see [4], p. 26) is the following theorem:

**Theorem 4** If $G$ is a graph with $\delta(G) > \lfloor |G| / 2 \rfloor$ then every two vertices of $G$ can be joined by a Hamiltonian path.

We shall deduce the following property for tight graphs.
Lemma 5 Every two vertices in a tight graph of order $n \geq 3$ can be joined by paths of order $n$ and $n - 1$.

Proof Let $G$ be a tight graph of order $n \geq 3$ and $u, v \in V(G)$. Theorem 4 implies that $u$ and $v$ may be joined by a path of order $n$. Select $w \neq u, v$ and consider $G' = G - w$. We have

$$\delta(G') \geq \delta(G) - 1 \geq |G|/2 > |G'|/2,$$

thus, again by Theorem 4, $u$ and $v$ may be joined by a path of order $n - 1$.

In [3] Bondy proved that every Hamiltonian graph $G$ of order $n$ with $\lfloor n^2/4 \rfloor$ edges contains $C_t$ for every $t \in [3, |G|]$ unless $n$ is even and $G$ is $K_{n/2,n/2}$. Since every graph $G$ of order $n$ with $\delta(G) \geq n/2$ is Hamiltonian, we obtain the following proposition.

Proposition 6 Every tight graph $G$ contains $C_t$ for every $t \in [3, |G|]$. \hfill $\square$

A theorem of Dirac [7] asserts that every 2-connected graph $G$ contains cycles of length at least $\min \{2\delta(G), |G|\}$; thus, the following proposition holds.

Proposition 7 Every loose 2-connected graph $G$ contains a cycle of length at least $2\delta(G) - 1$. \hfill $\square$

In [16] Voss and Zuluaga generalized the above theorem of Dirac by proving that every nonbipartite 2-connected graph $G$ of order $n$ contains an even cycle of length at least $\min \{n, 2\delta(G)\}$ and an odd cycle of length at least $\min \{n, 2\delta(G) - 1\}$. The following simple consequence holds.

Proposition 8 If $G$ is a loose, nonbipartite, 2-connected graph then $ec(G) \geq 2\delta(G)$ and $oc(G) \geq 2\delta(G) - 1$. \hfill $\square$

2.3 Decomposition into 3-connected graphs

The following technical lemma plays a crucial role in our arguments.
Lemma 9 For every integer \( k \geq 1 \) and real \( l \geq 0 \), there exists \( n_0 = n_0 (k, l) \) such that: if \( G \) is a graph of order \( n > n_0 \) and \( \delta (G) \geq n/k - l \) then there exists a set \( U \) of at most \( (2^k - 2) \) vertices such that all components of \( G - U \) are 3-connected.

Proof We shall use induction on \( k \). For every \( l \geq 0 \), let \( n_0 (1, l) = 2l + 2 \); for every integer \( k \geq 2 \) and \( l \geq 0 \), set

\[
n_0 (k, l) = kn_0 (k - 1, 2l + 1) + k (l + 1). \tag{1}
\]

Let us check that the lemma holds for \( k = 1 \). Suppose \( n > 2l + 2 \) and let \( G \) be a graph with \( \delta (G) \geq n - l \). Then every two vertices have at least \( 2 (n - l) - n = n - 2l > 2 \) neighbors in common and consequently cannot be separated by fewer than 3 vertices. Thus \( \kappa (G) \geq 3 \) and the lemma holds with \( U = \emptyset \).

Let now \( k > 1 \) and assume the lemma holds for all smaller \( k \) and all \( l \geq 0 \). If \( G \) is 3-connected, we set \( U = \emptyset \) and finish the proof, so assume \( 0 \leq \kappa (G) \leq 2 \) and set \( \kappa = \kappa (G) \).

Let \( U_0 \) be a set of \( \kappa \) cutvertices so that \( G - U_0 \) is a union of two vertex-disjoint graphs \( G_1 \) and \( G_2 \); set \( n_1 = |G_1|, \ n_2 = |G_2| \). We have \( \delta (G_1) \geq \delta - \kappa \) and

\[
n_1 \geq \delta (G_1) + 1 \geq \frac{n}{k} - l + 1 - \kappa. \tag{2}
\]

Therefore,

\[
n_2 = n - n_1 - \kappa \leq n - \frac{n}{k} + l - 1 = \frac{(k - 1) n}{k} + l - 1
\]

and so,

\[
\frac{n}{k} \geq \frac{n_2}{k - 1} - \frac{l - 1}{k - 1} \geq \frac{n_2}{k - 1} - l + 1,
\]

implying

\[
\frac{n}{k} - l - \kappa \geq \frac{n_2}{k - 1} - 2l - 1.
\]

By symmetry,

\[
\frac{n}{k} - l - \kappa \geq \frac{n_1}{k - 1} - 2l - 1.
\]

Thus, we find that

\[
\delta (G_1) \geq \delta - \kappa \geq \frac{n}{k} - l - \kappa \geq \frac{n_1}{k - 1} - 2l - 1.
\]
and, by symmetry, the same inequality holds for \( \delta(G_2) \). Observe that (1) and (2) imply

\[
n_1 \geq \frac{n}{k} - l + 1 - \kappa > n_0 (k - 1, 2l + 1).
\]

Applying the inductive assumption to the graph \( G_1 \), we find a set \( U_1 \subset V(G_1) \) such that \( |U_1| \leq 2^{k-1} - 2 \), and all components of \( G_1 - U_1 \) are 3-connected. By symmetry, we find a set \( U_2 \subset V(G_2) \) such that \( |U_2| \leq 2^{k-1} - 2 \) and all components of \( G_2 - U_2 \) are 3-connected. Setting \( U = U_0 \cup U_1 \cup U_2 \) we see that \( |U| \leq 2 (2^{k-1} - 2) + \kappa \leq 2^k - 2 \), completing the proof. \( \square \)

### 2.4 Preliminary results on paths and cycles

The following versatile theorem has been proved by Gould, Haxell, and Scott in [11]. It is implicit also in [14] but the authors of [14] dealt with a specific problem and failed to recognize the more general nature of the subject.

**Theorem 10** For all \( c > 0 \) there exists \( K = K(c) \) such that if \( n > 45Kc^{-4} \) and \( G = G(n) \) is a graph with \( \delta(G) \geq cn \) then \( C_t \subset G \) for all even \( t \in [4, ec(G) - K] \) and all odd \( t \in [K, oc(G) - K] \).

A minor modification of the original proof of this result in [11] is enough to generalize it in the following technical way.

**Theorem 11** For all \( c > 0 \) and \( q \) there exist \( n_0 = n_0(c, q) \) and \( K = K(c, q) \) such that if \( n > n_0 \) and \( G = G(n) \) is a graph with \( \delta(u) \geq cn \) for all but at most \( q \) vertices \( u \in V(G) \) then \( C_t \subset G \) for all even \( t \in [4, ec(G) - K] \) and all odd \( t \in [K, oc(G) - K] \).

Having the preliminary concepts in hand, the main idea of our proof of Theorem 1 can be stated as follows: if \( G \) is a loose graph, find a cycle of the desired parity of length much greater than \( \delta(G) \) and apply Theorem 11; if \( G \) is tight, the proof is completed by Proposition 6.

**Lemma 12** Suppose \( G \) is a 2-connected graph of order \( n \) with \( \delta(G) \geq cn \). Then, for every two vertices \( u \) and \( v \), there is a \( uv \)-path \( P \) satisfying \( 3 \leq |P| \leq \max \{ 3, 3/c \} \).
Proof Since $G$ is 2-connected, the vertices $u$ and $v$ are joined by a path of order at least 3. Among all $uv$-paths of order at least 3 select one of minimum order, say $P = u, w_1, ..., w_k, v$. The only possible chord in $P$ is the edge $uv$. Observe that no vertex in $G - P$ is joined to more than 3 vertices of $P$, or else there would be a $uv$-path shorter than $P$. Therefore, counting the cross edges $E(P, V \setminus P)$, we find that

$$3(n - k - 2) \geq e(P, V \setminus P) \geq \sum_{x \in P} (d(x) - 2) \geq (k + 2)(cn - 2).$$

Hence,

$$k + 2 \leq \frac{3n}{cn + 1} = \frac{3}{c + 1/n} < \frac{3}{c},$$

as claimed. \qed

A beautiful result of Andrásfai, Erdős and Sós in [1] states that, if $G$ is a nonbipartite graph of order $n$ and $\delta(G) > \frac{2n}{2k+1}$, then $G$ contains a cycle $C_{2l-1}$ for some $l \leq k$. Below is a slight variation of this theorem.

Lemma 13 Let the real number $c$ satisfy $0 < c < 1$. Suppose $G$ is a nonbipartite graph of order $n \geq 2/c$ and $d(u) \geq cn$ for all but $k$ vertices of $G$. Then $G$ contains an odd cycle of length at most $\max \{2/c + k, 3\}$.

Proof Select an odd cycle $C$ of minimum length; clearly $C$ contain no chords, and we may assume $|C| > 3$. Each vertex in $V \setminus C$ is joined to at most 2 vertices from $C$, or else there would be an odd cycle shorter than $C$. Hence, estimating the cross-edges $E(C, V \setminus C)$, we find that

$$(n - |C|) 2 \geq e(C, V \setminus C) \geq \sum_{u \in C} d(u) - 2|C| \geq (|C| - k)cn + 2k - 2|C|,$$

implying

$$\frac{2}{c} + k > |C|,$$

as claimed. \qed

Lemma 14 For all positive real $c < 1$ there exist $l = l(c)$, $m = m(c)$ and $n_0 = n_0(c)$ such that if $G$ is a loose 3-connected graph of order $n > n_0$, and $\delta(G) = \delta \geq cn$, then, either

(a) $G$ contains odd and even cycles of length at least $2\delta - l$, or

(b) for every two vertices $u, v$ there is a $uv$-path of length greater than $\frac{4}{3}\delta - m$. 

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Proof Indeed, let

\[ k = \lceil 1/c \rceil, \]
\[ l = l(c) = 2^{k+1} + 6k, \]
\[ m = m(c) = \frac{4}{3}(2^k + 3k), \]

and fix two arbitrary vertices \( u, v \); applying Lemma 12, we find a \( uv \)-path \( P \) with \( |P| \leq 3k \). Let \( G' = G - P \).

Case 1: \( G' \) is tight

Since \( G \) is 2-connected, there exist two vertex-disjoint paths \( P_1 = u, ..., u' \) and \( P_2 = v, ..., v' \), such that \( P_1 \cap G' = \{u'\} \), \( P_2 \cap G' = \{v'\} \). Applying Lemma 5, we can join \( u' \) and \( v' \) by a Hamiltonian path within \( G' \). Thus, we see that \( u \) and \( v \) are joined by a path of order at least \( |G'| + 2 \). Since \( G \) is loose, we have

\[ 2\delta \leq n + 1 \leq |G'| + 3k + 1; \]

hence,

\[ |G'| + 2 \geq 2\delta - 3k + 1 > \frac{4}{3}\delta - m \]

for \( n \) sufficiently large. Therefore, (b) holds, completing the proof in this case.

Case 2: \( G' \) is loose

Assuming that \( n \) is large enough and applying Lemma 9, we can remove a set \( U \subset V(G') \) of at most \( 2^k - 2 \) vertices so that all components of \( G'' = G' - U \) are 3-connected.

If \( G'' \) is tight then, by Proposition 6, \( G \) contains all cycles up to \( n' = n - |P| - |U| \). Since \( G \) is loose,

\[ 2\delta - l \leq n + 1 - l = n - 2^{k+1} - 6k + 1 < n - |P|-|U| = n', \]

and (a) follows, completing the proof.

Hereafter we shall assume that \( G'' \) is loose. To complete the proof we shall consider two cases - (i) \( G'' \) has two tight components, and (ii) \( G'' \) has a loose component.

Case 2.i: \( G'' \) has two tight components

Let \( G''_1 \) and \( G''_2 \) be two tight components of \( G'' \). Since \( G \) is 2-connected, there exist two vertex-disjoint paths \( P_1 = x, ..., x' \) and \( P_2 = y, ..., y' \), such that

\[ P_1 \cap G''_1 = \{x\}, \quad P_1 \cap G''_2 = \{x'\}, \quad P_2 \cap G''_1 = \{y\}, \quad P_2 \cap G''_2 = \{y'\}. \]
Applying Lemma 5, join $x$ and $y$ by a Hamiltonian path $H_1$ within $G''_0$; join $x'$ and $y'$ by a Hamiltonian path $H_2$ within $G''_2$. The paths $P_1, H_1, P_2, H_2$ form a cycle $C$ of length at least

$$|H_1| + |H_2| \geq \delta (G''_0) + \delta (G''_2) + 2 \geq 2\delta - 2|P| - 2|U| + 2 \geq 2\delta - 2^{k+1} - 6k + 6 > 2\delta - l.$$ 

Similarly, applying Lemma 5, we can obtain a cycle of length $|C| - 1$; hence $(a)$ holds, completing the proof in this case.

Case 2.ii: $G''$ has a loose component

Let $G_0$ be a loose component of $G''$. By Proposition 7, $G_0$ contains a cycle $C$ of length at least

$$2\delta (G_0) - 1 \geq 2 (\delta - 2^k - 3k + 2) - 1 > 2 (\delta - 2^k - 3k).$$

Since $G$ is 3-connected, there exist three vertex-disjoint paths $P_1 = x, \ldots, x', P_2 = y, \ldots, y'$, and $P_3 = z, \ldots, z'$ such that

$$P_1 \cap P = \{x\}, \quad P_1 \cap C = \{x'\},$$
$$P_2 \cap P = \{y\}, \quad P_2 \cap C = \{y'\},$$
$$P_3 \cap P = \{z\}, \quad P_3 \cap C = \{z'\}.$$ 

Clearly two of the vertices $x', y', z'$ (say $x'$ and $y'$) are joined along $C$ by a path $P_4$ of order at least $2|C|/3$. Therefore $P_1, P_4, P_2$ form an $xy$-path $Q$ vertex-disjoint from $P$, except for the vertices $x$ and $y$, and with

$$|Q| > \frac{2}{3} |C| > \frac{4}{3} (\delta - 2^k - 3k).$$

Clearly $Q$ can be extended to a $uv$-path at least as long as $Q$. Therefore $(b)$ holds, completing the proof of Lemma 14.

The following theorem contains the bulk of the proof of Theorem 1.

**Theorem 15** For all integer $q \geq 0$ and positive real $\alpha < 1$ there exists $Q = Q(\alpha, q)$ such that the following holds:

Let $G$ be a 2-connected nonbipartite graph of order $n$ with $d(u) \geq \delta \geq \alpha n$ for all but at most $q$ vertices $u$ of $G$. Then, for $n$ sufficiently large, $C_t \subset G$ for every $t \in [Q, \delta + 1]$. 

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Proof If \( G \) is tight the assertion follows from Proposition 6, so we shall assume that \( G \) is loose. Our main tool to the end of the proof will be Theorem 11; for its efficient application it is enough to prove that \( G \) contains sufficiently large odd and even cycles.

Applying Lemma 13, select an odd cycle \( C \) of length at most \( \max \left\{ \frac{2}{\alpha} + q, 3 \right\} \leq \frac{2}{\alpha} + q + 1. \)

Write \( W \) for the set of the vertices of \( C \) together with all vertices \( u \in V(G) \) such that \( d(u) < \delta \). Clearly,

\[ |W| \leq \frac{2}{\alpha} + 2q + 1. \]

Let \( G' = G - W \).

Case 1: \( G' \) is tight

By Proposition 6, \( G \) contains all cycles up to \( n' = n - |W| \). The desired result follows immediately if \( n' > \delta \). But \( \delta \geq n' \) is impossible for \( n \) sufficiently large, because \( G \) is loose.

Case 2: \( G' \) is loose

Observe that

\[ \delta (G') \geq \delta - |W| \geq \alpha n - \frac{2}{\alpha} - 2q - 1 \geq \alpha n' - \frac{2}{\alpha} - 2q - 1. \]

Applying Lemma 9 with \( k = \lceil 1/\alpha \rceil \) and \( l = \lceil 2/\alpha \rceil + 2q + 1, \) for \( n \) sufficiently large, remove a set \( U \) of at most \( (2^k - 2) \) vertices such that all components of the graph \( G'' = G' - U \) are 3-connected.

If \( G'' \) is tight then, by Proposition 6, \( G \) contains all cycles up to \( n' = n - |W| - |U| \). Since \( G \) is loose, \( \delta < n' \) for \( n \) sufficiently large and the desired result follows.

Hereafter we shall assume that \( G'' \) is loose. To complete the proof we shall consider two cases - (i) \( G'' \) has two tight components, and (ii) \( G'' \) has a loose component.

Case 2.i: \( G'' \) has two tight components

Let \( G''_1 \) and \( G''_2 \) be two tight components of \( G'' \). Since \( G \) is 2-connected, there exist two vertex-disjoint paths \( P_1 = x, ..., x' \) and \( P_2 = y, ..., y' \), such that

\[ P_1 \cap G''_1 = \{ x \}, \quad P_1 \cap G''_2 = \{ x' \}, \quad P_2 \cap G''_1 = \{ y \}, \quad P_2 \cap G''_2 = \{ y' \}. \]
Applying Lemma 5, join \( x \) and \( y \) by a Hamiltonian path \( H_1 \) within \( G''_1 \); join \( x' \) and \( y' \) by a Hamiltonian path \( H_2 \) within \( G''_2 \). The paths \( P_1, H_1, P_2, H_2 \) form a cycle of length at least

\[
|H_1| + |H_2| \geq \delta (G''_1) + \delta (G''_2) + 2 \geq 2\delta - 2 |W| - 2 |U| + 2
\geq 2\delta - 2^{k+1} - 2l + 6 = 2\delta - 2^{[1/\alpha]+1} - 4 \left\lceil \frac{1}{\alpha} \right\rceil - 4q + 8.
\]

Similarly, applying Lemma 5, join \( x \) and \( y \) within \( G''_1 \) by a path \( H'_1 \) of order \( |G''_1| - 1 \). The paths \( P_1, H'_1, P_2, H_2 \) form a cycle of length one less than the previous one. Therefore, we have

\[
e c (G) \geq 2\delta - 2^{[1/\alpha]+1} - 4 \left\lceil \frac{1}{\alpha} \right\rceil - 4q + 8
\]

\[
o c (G) \geq 2\delta - 2^{[1/\alpha]+1} - 4 \left\lceil \frac{1}{\alpha} \right\rceil - 4q + 7,
\]

Theorem 11 implies that \( G \) contains a cycle \( C_t \) for every

\[
t \in \left[ K, 2\delta - 2^{[1/\alpha]+1} - 4 \left\lceil \frac{1}{\alpha} \right\rceil - 4q + 7 - K \right],
\]

where \( K \) depends on \( q \) and \( \alpha \), completing the proof in this case.

Case 2.ii: \( G'' \) has a loose component

Let \( G_0 \) be a loose component of \( G'' \); by the construction of \( G'' \), \( G_0 \) is 3-connected. Notice that,

\[
\delta (G_0) \geq \delta - |W| - |U| \geq \delta - \frac{2}{\alpha} - 2q - 2^{[1/\alpha]} + 1.
\]

Setting \( c = 15\alpha/16 \) we see that, for \( n \) sufficiently large,

\[
\delta (G_0) > cn \geq c |G_0|
\]

Applying Lemma 14 to \( G_0 \), we find that either \( G_0 \) contains odd and even cycles of length at least \( 2\delta (G_0) - l (15\alpha/16) \), or for every two vertices \( u, v \) of \( G_0 \), there is a \( uv \)-path of length greater than

\[
\frac{4}{3} \delta (G_0) - m (15\alpha/16) \geq \frac{4}{3} \delta (G) - M,
\]

where

\[
M = m (15\alpha/16) + \frac{2}{\alpha} + 2q + 2^{[1/\alpha]} - 1
\]

is independent of \( n \).
In the first case we complete the proof applying Theorem 11, so suppose the second case holds. Since $G$ is 2-connected, there exist two vertex-disjoint paths $P_1 = x, \ldots, x'$ and $P_2 = y, \ldots, y'$, such that

$P_1 \cap C = \{x\}, \ P_1 \cap G_0 = \{x'\}, \ P_2 \cap C = \{y\}, \ P_2 \cap G_0 = \{y'\}$.

Join $x'$ and $y'$ within $G_0$ by a $x'y'$-path $H_1$ of order at least $4\delta(G)/3 - M$. Taking $H_2$ to be one of the two $xy$-paths along the cycle $C$ we see that the paths $P_1, H_1, P_2, H_2$ form a cycle of length at least $4\delta(G)/3 - M$, and so,

$$ec(G) \geq \frac{4}{3} \delta(G) - M, \quad oc(G) \geq \frac{4}{3} \delta(G) - M.$$ 

Hence, again applying Theorem 11, the proof is completed. \hfill \Box

Since every bipartite graph $G$ contains a cycle of length at least $2\delta(G)$, Theorem 11 implies the following result.

**Lemma 16** For all $k > 1$ there exist $n_0 = n_0(k)$ and $K = K(k)$ such that: if $G$ is a bipartite graph of order $n > n_0$ and $\delta(G) \geq n/k$ then $C_l \subseteq G$ for every even $l \in [4, 2\delta - K(k)]$.

To prove the existence of short odd cycles we shall need the following theorem from [12].

**Theorem 17** Let $l, m$ be positive integers. There exist $n_0 = n_0(l, m)$ and $c = c(l, m)$ such that for every $G$ of order $n > n_0$ and minimum degree

$$\delta(G) > \frac{n}{4l + 2} + c,$$

if $C_{2t+1} \subseteq G$, for some $l \leq t \leq 4l + 1$ then $C_{2t+2j+1} \subseteq G$ for every $j \in [m]$. \hfill \Box

We shall use this theorem to derive the following lemma.

**Lemma 18** Let $k \geq 2$ be integer and $G$ be a nonbipartite graph of order $n$. Suppose $\delta(G) \geq n/k$ and $G \notin B_k(n)$. Then for every integer $s > 0$ and $n$ sufficiently large, $G$ contains a cycle $C_t$ for all odd $t \in [2k - 1, 2k - 2 + s]$. 

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Proof Since \( G \notin B_k(n) \), \( C_{2l+1} \subset G \) for some integer \( l \) satisfying \( 2l+1 > k/2 \); select \( l \) to be the minimum integer with this property. A result of Håggkvist ([13], Theorem 2) states that if \( \delta(G) > 2n/(2k+1) \) and \( G \) contains cycles of length greater than \( k/2 \) then \( C_{2k-1} \subset G \) for \( n \) sufficiently large. Since in our case, \( G \notin B_k(n) \) it follows that \( G \) has an odd cycle longer than \( k/2 \). Hence, since \( \delta(G) \geq n/k > 2n/(2k+1) \), we conclude that \( C_{2k-1} \subset G \), and so \( 2l+1 \leq 2k-1 \). On the other hand, from \( 2l+1 > k/2 \) it follows that
\[
4l + 1 \geq k,
\]
and therefore,
\[
\delta(G) \geq \frac{n}{4l+1} > \frac{n}{4l+2} + c
\]
for large \( n \). The assertion now follows from Theorem 17.

2.5 Proof of the Theorem 1

Proof of Theorem 1 Notice first that, by a result of Bondy and Simonovits [5], \( G \) contains \( C_t \) for every even \( t \geq 4 \) and \( n > n_0(t) \). Similarly, if \( G \) is nonbipartite, by Lemma 18, \( G \) contains \( C_t \) for every odd \( t \geq 2k - 1 \) and \( n > n_0(t) \). Therefore, we need to prove the theorem for sufficiently large cycles.

We may assume that \( G \) is nonbipartite - otherwise assertion \( i) \) follows from Lemma 16.

Suppose that \( G \) is decomposed into blocks. Note that every endblock \( B \) has a vertex \( u \) that is not a cutvertex, so \( \Gamma(u) \subset V(B) \), so every endblock has at least \( \delta(G)+1 \) vertices.

From Lemma 18 we know that \( G \) contains an odd cycle \( C \) of length \( |C| > k \); let \( G_1 \) be the block containing \( C \). We shall prove that \( G_1 \) has fewer than \( k \) cutvertices. Observe first that each cutvertex \( u \in V(G_1) \) separates some endblock \( B(u) \) from \( G_1 \setminus \{u\} \); thus \( B(u) \cap B(v) = \emptyset \) for every two distinct cutvertices \( u, v \in V(G_1) \). Since every endblock has at least \( \delta(G)+1 \) vertices, writing \( q \) for the number of all cutvertices in \( G_1 \), we see that
\[
n \geq q(\delta(G)+1) > qn/k,
\]
and consequently \( q < k \).

In particular, \( G_1 \) has a vertex \( w \) that is not a cutvertex; we see that \( \Gamma(w) \subset V(G_1) \), and so the order of \( G_1 \) increases with \( n \).
Since $d(w) \geq \delta(G)$ for all $w \in V(G_1)$ that are not cutvertices, applying Theorem 15 to the graph $G_1$ with $\alpha = 1/k$ and $\delta = \delta(G)$, we deduce that $C_t \subset G_1 \subset G$ for every $t \in [Q(1/k, q), \delta + 1]$. The proof is completed. \hfill \Box

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